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The motion of tethered tug–debris system with fuel residuals

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Abstract

Active debris removal using a space tug with a tether is one of the promising techniques to decrease the population of large non-functional satellites and orbital stages in near Earth orbits. Properties of debris should be taken into account in the development of the space tugs. In this paper we consider the motion of a debris objects with fuel residuals that can affect the safety of the debris transportation process. The equations of the attitude motion of the tug–debris system in a central gravitational field are derived. Stationary solutions of the equations are found. The system of linearized equations are introduced that can be used for short term analysis. The numerical simulation results are provided that show good accuracy of the linearized equations. Proposed equations can be used to analyze the attitude motion of the tug–debris system and to determine the conventional parameters for safe tethered transportation of space debris.

Keywords: Space debris; Tether; Space tug; Fuel residuals

1. Introduction and problem formulation

During the last years several active debris removal methods were developed (Dafu and Xianren, 2008; Forward et al., 2000; Kitamura et al., 2014). Active debris removal using tethered space debris is one of the promising techniques to decrease the population of large non-functional satellites and orbital stages on the near Earth orbit (Jasper et al., 2012; Jasper and Schaub, 2014). Reference (Bonnal et al., 2013) noted that there are two types of large space debris: spacecraft and orbital stages. On the one hand, orbital stages may be more easily deorbited because they don't have large appendages such as antennas and solar panels. Tethered transportation of spacecraft with flexible appendages is considered in Aslanov and Yudintsev (2014), Aslanov and Yudintsev (2014). In the other hand orbital stages may contain fuel residuals that affect to the deorbit process. The effect of liquid fuel slosh on spacecraft has been explored in the literature (Reyhanoglu and Rubio Hervas, 2012; Yue, 2011; Rubio Hervas and Reyhanoglu, 2014). This literature considers the control of vehicle with fuel slosh dynamics. The aim of the present paper is to develop a simple mathematical model of the tug–debris system with fuel in terms of a multibody system model that can be used to analyze active debris removal missions.

The paper is divided into four parts. In the Section 2 nonlinear equations of the system are obtained that describe the motion of tug–debris system in a central gravitational field. The attitude motion near the stationary point is considered in the Section 3. Simplified linear equations are derived. The numerical simulation results provided in the Section 4. These results show that simplified equations give a good approximation to the exact solution.

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2. Mathematical model

This section formulates the dynamics of the tug–debris system. The debris is represented as a rigid body with fuel sloshing mass. We suppose that the space tug has an attitude control system that maintains required orientation of the tug, so the tug is represented as a mass point. The tether is considered as a massless rod. We use Lagrange formalism to derive the equations of the relative motion. A scheme of the system is shown in Fig. 1.

A solution of the general problem of oscillations of residual fuel in a container is extremely difficult. Here we use the simplest model where the sloshing liquid is modeled as an equivalent pendulum model. Reference (Peterson et al., 1989) demonstrates that equivalent pendulum model can approximate motion of the fuel residuals (Reyhanoglu, 2010). This model can be used when the oscillations of liquid are small (Abramson and Silverman, 1966; Ibrahim, 2005).

2.1. Kinematics of the system

The plain motion of the tug–debris system is considered in orbital frame Cx_oy_o attached to the center of mass of the entire system C. The motion of the system occurs throughout the action of the thrust $\boldsymbol{F}$ and a central gravity force. The thrust $\boldsymbol{F}$ is assumed to act along the axis Cx_o.

The position of the debris relative to Cx_oy_o frame is described by the angle $\theta + \alpha$ and the vector $\boldsymbol{R}_2$. The angle α defines the orientation of the tether. The angle θ is an angle between the tether and the longitudinal axis of the debris (Fig. 2). The tether length is l_1. The tether attachment point is determined by the vector $\boldsymbol{\rho}_1$. We suppose that the tether is attached at the longitudinal axis of debris $\boldsymbol{\rho}_1 = \{x_1, 0\}^T$. The attachment point of the equivalent pendulum is defined by the column vector $\boldsymbol{\rho}_3 = \{x_3, 0\}^T$. The pendulum length is l_3. The angle φ of the pendulum with respect to the debris longitudinal axis representing the fuel slosh (Fig. 2). The column vector $\mathbf{R}_2$ denotes the position of the center of mass (C_2) of the debris relative to the frame Cx_oy_o. The position column vector of the tug is expressed as

$$\mathbf{R}_1 = \mathbf{R}_2 + \boldsymbol{A}(\theta + \alpha) \cdot \boldsymbol{\rho}_1 + \mathbf{A}(\alpha) \cdot \mathbf{e}_x l_1 \tag{1}$$

where $\mathbf{e}_x = [1, 0]^T$ is an unit column vector, $\boldsymbol{A}(\alpha), \boldsymbol{A}(\theta + \alpha)$ are rotation matrices

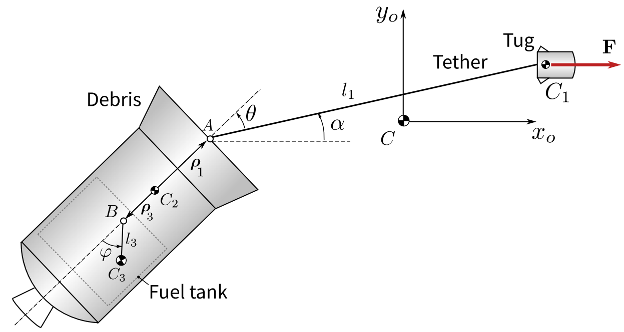

Fig. 2. The space debris and the space tug.

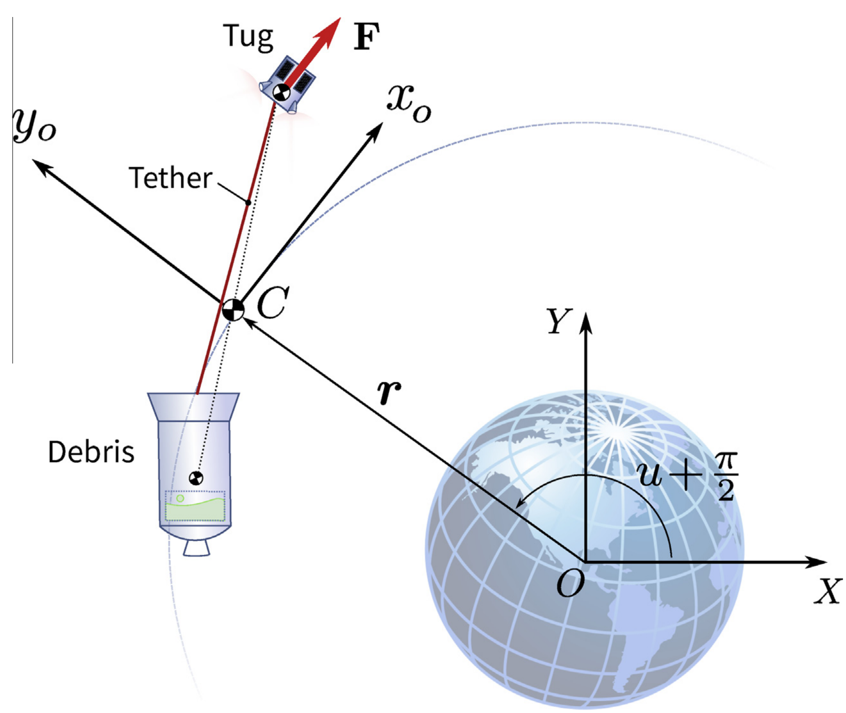

Fig. 1. Orbital frame.

$$\boldsymbol{A}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

$$\boldsymbol{A}(\theta+\alpha) = \begin{bmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) \end{bmatrix}$$

Position of the pendulum (fuel lump) relative to the frame Cx_oy_o can be computed as

$$\mathbf{R}_3 = \mathbf{R}_2 + \mathbf{A}(\theta+\alpha)\cdot(\boldsymbol{\rho}_3 - \mathbf{A}(\varphi)\cdot\mathbf{e}_x l_3) \tag{2}$$

where

$$\boldsymbol{A}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

The motion of tug, debris and the fuel is considered relative to the center of mass of the system therefore we can write

$$\sum_{i=1}^{3}\mathbf{R}_i m_i = 0 \tag{3}$$

where m_i is a mass of the tug ($i = 1$), space debris ($i = 2$) and fuel ($i = 3$). Using (3) we can eliminate $\boldsymbol{R}_2$ from the expressions (1) and (2).

2.2. Kinetic energy and Lagrange equations

The kinetic energy of the relative motion of the tug debris and the fuel slug is given by

$$2T = \sum_{i=1}^{3} m_i\mathbf{V}_i^2 + J_z(\dot\theta+\dot\alpha)^2 \tag{4}$$

where J_z is the moment of inertia of the debris. We suppose we know all the moments inertia of the debris J_x, J_y, J_z and $J_y = J_z$; $\boldsymbol{V}_i = d\boldsymbol{R}_i/dt$ is the velocity of the body i:

$$\boldsymbol{V}_1 = \frac{1}{M}\begin{bmatrix} l_1\dot\alpha(m_1-M)s_\alpha - l_3m_3\omega_3 s_{\alpha+\theta+\varphi} + \omega_2(x_1(m_1-M)+m_3x_3)s_{\alpha+\theta} \\ l_1\dot\alpha(M-m_1)c_\alpha + l_3m_3\omega_3 c_{\alpha+\theta+\varphi} + \omega_2(x_1(M-m_1)-m_3x_3)c_{\alpha+\theta} \end{bmatrix}$$

$$\boldsymbol{V}_2 = \frac{1}{M}\begin{bmatrix} l_1m_1\dot\alpha s_\alpha - l_3m_3\omega_3 s_{\alpha+\theta+\varphi} + \omega_2(m_1x_1+m_3x_3)s_{\alpha+\theta} \\ -l_1m_1\dot\alpha c_\alpha + l_3m_3\omega_3 c_{\alpha+\theta+\varphi} - \omega_2(m_1x_1+m_3x_3)c_{\alpha+\theta} \end{bmatrix}$$

$$\boldsymbol{V}_3 = \frac{1}{M}\begin{bmatrix} m_1(l_1\dot\alpha s_\alpha + x_1\omega_2 s_{\alpha+\theta}) - (M-m_3)(x_3\omega_2 s_{\alpha+\theta} - l_3\omega_3 s_{\alpha+\theta+\varphi}) \\ -l_1m_1\dot\alpha c_\alpha - l_3\omega_3(M-m_3)c_{\alpha+\theta+\varphi} - \omega_2 c_{\alpha+\theta}(x_3(m_3-M)+m_1x_1) \end{bmatrix}$$

where

$$\omega_2 = \dot\alpha + \dot\theta, \quad \omega_3 = \omega_2 + \dot\varphi$$

The letters c and s with a variable or an expression in the subscript denote the cosine and sine of variable or expression respectively. Gravitational force $\boldsymbol{G}_i$ that act on the body i is

$$\mathbf{G}_i = -\frac{\mu m_i}{\mathbf{r}_i^3}\mathbf{r}_i \tag{5}$$

where μ is the Earth's gravitational parameter, $\mathbf{r}_i$ is the position vector of the body i relative to the Earth's center

$$\boldsymbol{r}_i = \boldsymbol{r} + \boldsymbol{R}_i, \; i = 1,2,3 \tag{6}$$

$\boldsymbol{r} = \{0, r\}^T$ is the position column vector of the system center of mass. The column vector of the thrust force is $\boldsymbol{F} = \{F, 0\}^T$. We assume $F = \text{const}$.

The orbital frame Cx_0y_0 is a non inertial frame so we have to add inertial forces

$$\boldsymbol{\Phi}_i = -m_i(\boldsymbol{a}_o + \boldsymbol{\omega}_o \times (\boldsymbol{\omega}_o \times \boldsymbol{R}_i) + \boldsymbol{\varepsilon}_o \times \boldsymbol{R}_i + 2\boldsymbol{\omega}_o \times \boldsymbol{V}_i) \tag{7}$$

where $\boldsymbol{a}_o$ is acceleration of the system's center of mass

$$\boldsymbol{a}_o = \begin{bmatrix} -r\ddot\nu - 2\dot r\dot\nu \\ \ddot r - r\dot\nu^2 \end{bmatrix} \tag{8}$$

$\boldsymbol{\omega}_o, \boldsymbol{\varepsilon}_o$ are the angular velocity and the angular acceleration of the frame Cx_0y_0

$$\boldsymbol{\omega}_o = \{0, 0, \dot\nu\}^T, \quad \boldsymbol{\varepsilon}_o = \{0, 0, \ddot\nu\}^T, \tag{9}$$

To get derivatives $\dot r, \dot\nu, \ddot r, \ddot u$ let us write the equations for the center of mass of the system in osculating elements ν, p, e, σ (Okhotsimskii, 1964):

$$\dot\nu = \sqrt{\mu(1+e\cos\nu)}, \tag{10}$$

$$\dot p = -2\frac{pF}{mv}, \tag{11}$$

$$\dot e = -2\frac{e+\cos\nu}{mv}F, \tag{12}$$

$$\dot\sigma = -2\frac{\cos\nu}{mve}F \tag{13}$$

where $\nu = u - \sigma$ is a true anomaly angle, p is the focal parameter of the orbit, e is the eccentricity, σ is the angle of periapsis, v is the orbital velocity

$$v = \sqrt{\frac{\mu(1+e^2+2e\cos\nu)}{p}}, \tag{14}$$

r is a distance of the center of mass of the system relative to the center of the Earth

$$r = \frac{p}{1+e\cos\nu}. \tag{15}$$

The derivatives $\dot r, \dot\nu, \ddot r, \ddot u$ can be obtained after differentiating the Eqs. (13) and (15).

The space debris is considered as a rigid body, therefore a gravitational torque should be taken into account (Beletskii, 1966)

$$M_{2z} = \frac{3\mu}{2r_2^3}(J_y - J_x)\sin 2(\theta+\alpha) \tag{16}$$

The space tug and the fuel lump are considered as point masses, so $M_{1z} = M_{3z} = 0$.

Now we can write generalized forces (Taylor, 2005)

$$Q_k = \sum_{i=1}^{3}(\boldsymbol{G}_i + \boldsymbol{\Phi}_i)\cdot\frac{\partial\boldsymbol{R}_i}{\partial q_k} + \boldsymbol{F}\cdot\frac{\partial\boldsymbol{R}_1}{\partial q_k} + M_{kz}, \quad k = 1,2,3. \tag{17}$$

where q_k is a generalized coordinate

$$q_1 = \theta, \; q_2 = \alpha, \; q_3 = \varphi \tag{18}$$

Using (4) and (17) we can build the differential equations of the considered system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i, \quad i = 1,2,3 \tag{19}$$

2.3. Simplified equations

The obtained Eq. (19) can be integrated but they are very cumbersome and inconvenient for motion analysis. To determine the stationary solutions and then to study of the motion near the stationary point, let us write simplified equations taking the following assumptions.

- the orbit of the mass center is not changed $r = \text{const}$, but we take into account the non inertial motion of the orbital frame,
- the tether length is small relative to r.

The inertial forces can be simplified as follows. The centrifugal force is

$$\mathbf{\Phi}_{wi} = -m[\boldsymbol{a}_o + \boldsymbol{\omega}_o \times (\boldsymbol{\omega}_o \times \boldsymbol{R}_i)] \tag{20}$$

where $\omega_o = \sqrt{\mu/r^3} = \text{const}$ is the orbital angular velocity, $\boldsymbol{a}_o$ is a acceleration of the mass center

$$\boldsymbol{a}_o = -\omega_o^2 \mathbf{r} \tag{21}$$

The second term of (20) can be rewritten as

$$\boldsymbol{\omega}_o \times (\boldsymbol{\omega}_o \times \mathbf{R}_i) = -\omega_o^2 \mathbf{R}_i,$$

so the centrifugal force get the form

$$\mathbf{\Phi}_{wi} = m\omega_o^2(\mathbf{r} + \mathbf{R}_i) \tag{22}$$

For the circular orbit

$$\omega_o^2 = \frac{\mu}{r^3}, \tag{23}$$

so the expression (5) get the form

$$\mathbf{G}_i = -\frac{\mu m_i}{r_i^3}\mathbf{r}_i = -\omega_o^2 m_i \left[\frac{r}{r_i}\right]^3 \mathbf{r}_i \tag{24}$$

The expression in the braces for $R_i \ll r$ can be written

$$\left[\frac{r}{r_i}\right]^3 \approx \left[1 + \frac{2y_i}{r}\right]^{-3/2} \approx 1 - 3\frac{y_i}{r}$$

This simplification allows to rewrite $\boldsymbol{G}_i$

$$\mathbf{G}_i \approx -\omega_o^2 m_i \left[1 - 3\frac{y_i}{r}\right]\mathbf{r}_i \tag{25}$$

The generalized forces get the form

$$Q_k = \boldsymbol{F}\cdot\frac{\partial \boldsymbol{R}_1}{\partial q_k} + \sum_{i=1}^{3} m_i\left(-\frac{\boldsymbol{F}}{M} + 3\omega_o^2\frac{y_i}{r}\boldsymbol{r} - 2\boldsymbol{\omega}_o \times \boldsymbol{v}_i\right)\cdot\frac{\partial \boldsymbol{R}_i}{\partial q_k} + Q_k^*, \quad k = 1,2,3 \tag{26}$$

where $\boldsymbol{\omega}_o = \{0,0,\omega_o\}^T, \boldsymbol{r} = \{0,r,0\}; Q_2^* = Q_3^* = 0,$

$$Q_1^* = \frac{3\mu}{r^3}(J_z - J_x)\cos(\theta+\alpha)\sin(\theta+\alpha) - J_z\ddot{v} \tag{27}$$

New generalized forces allow to write the simplified equations of the motion, which will be used to study an evolution of the system around stationary points. The equations are given in the Appendix A.

3. Motion of the system near the stationary point

3.1. Stationary solution

Considered tug-tether–debris system can be represented as a two mass system connected with a massless rod. In central gravitational field this two mass system has two stationary points $\alpha_{01} = 0$ (unstable) and $\alpha_{02} = \pi/2$ (stable) (Beletskii, 1966). Tug's thrust F shifts stable stationary point to $\alpha_{02} < \pi/2$. This stationary point depends on the tug's thrust, length of the tether and masses of the tug and the debris.

To determine the stationary solutions of the equations, the derivatives are set to zero

$$\dot{\theta} = \dot{\varphi} = \dot{\alpha} = 0, \quad \ddot{\theta} = \ddot{\varphi} = \ddot{\alpha} = 0 \tag{28}$$

In this case, the system of Eqs. (19) converted to a non-linear system of equations for the unknown angles $\theta_0, \alpha_0, \varphi_0$.

To simplify the search for solutions of this system let us obtain an approximation to get the stationary solution of the system. We equate Q_α (26) to zero and set $\theta = 0$ and $\varphi = 0$. We get

$$(a\cos\alpha - Fb)\sin\alpha = 0 \tag{29}$$

where a, b are coefficients that depend on the parameters of the system

$$a = 3\omega_0^2\Big[(x_1 + l_1)^2 M_1 m_1 + (x_3 - l_3)^2 M_3 m_3 + 2m_1 m_3(l_3 - x_3)(l_1 + x_1)\Big]$$

$$b = M_1(x_1 + l_1) + m_3(l_3 - x_3)$$

where $M = m_1 + m_2 + m_3, M_i = M - m_i, i = 1,2,3.$

Fig. 3a demonstrates the generalized force Q_α as a function of α. The figure shows two stationary points $\alpha_{01} = 0$ and $\alpha_{02} \approx 0.35$. Fig. 3 is plotted for parameters presented in the Table 1. Eq. (29) shows that the system has two stationary points. The first stationary point is determined by the condition $\sin\alpha = 0$. The second stationary solution is determined by the condition

$$a\cos\alpha - Fb = 0 \tag{30}$$

that exists only if $Fb/a \leqslant 1$. Fig. 3b shows stationary solution α_0 as a function of tether length l_1 for $F = 0.3$ N and the parameters of the system that are presented in the Table 1. Fig. 3b demonstrates that there is only one stationary solution for the tether length $l_1 < l_1^* \approx 400$ m

$$\alpha_0 = 0, \quad \theta_0 = 0. \tag{31}$$

There are two stationary solutions for $l_1 > l_1^*$

$$\alpha_{01} = 0,\ \theta_{01} = 0 \text{ and } \alpha_{02} > 0,\ \theta_{02} > 0. \tag{32}$$

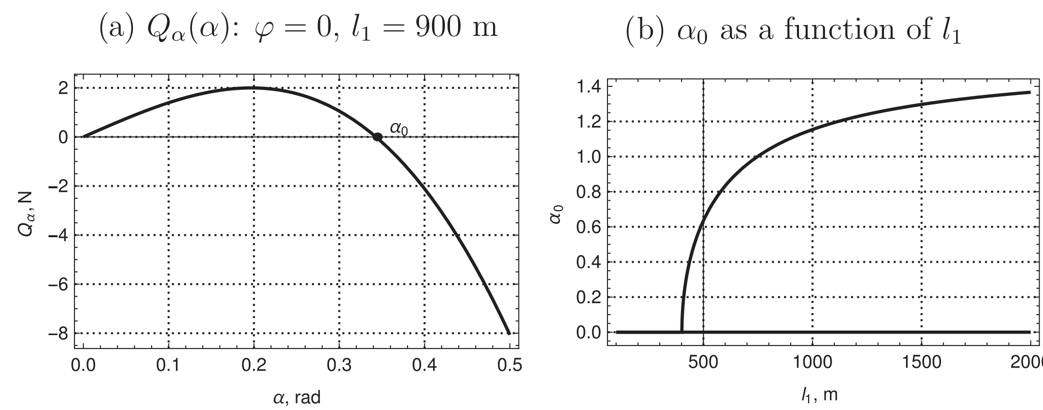

Fig. 3. Generalized force Q_α as a function of θ and stationary point α_0 as a function of the tether length.

Table 1
Parameters of the space tug and the debris.

Parameter	Value	Parameter	Value
m_1, kg	200	m_2, kg	3000
m_3, kg	500	l_3, m	1
x_3, m	−1	x_1, m	5
$J_y = J_z$, kg m^2	10000	J_x, kg m^2	3000

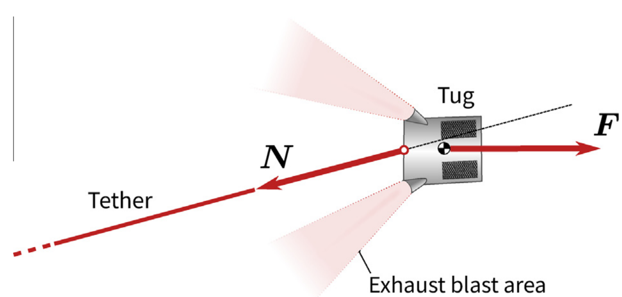

Fig. 4. Exhaust blast area of the space tug.

To reduce the influence of the tether force to the motion of the space tug tether length should be taken that results only one stationary solution $\alpha = 0$. In this case, the disturbing torque of the tether force $\boldsymbol{N}$, acting to the tug is minimal. Also the motion with a small angle $\alpha \approx 0$ reduces risk of tether rupture due exposure to the tug's jet Fig. 4.

3.1.1. Linearized equations

After determining the stationary solutions we can construct linearized equations of motion in the neighborhood of the stationary point $\theta_0 = \alpha_0 = \varphi_0 = 0$.

We rewrite the kinetic energy of the system as a quadratic form with constant coefficients

$$T(q, \dot{q}) \approx T(q_0, \dot{q}) \tag{33}$$

where $q_0 = (\theta_0, \alpha_0, \varphi_0)$ is the stationary point. In the expressions for the generalized forces we set $q = q_0 + \hat{q}$, were $\hat{q} = (\hat{\theta}, \hat{\alpha}, \hat{\varphi})$ are new variables that describe the deviation from the stationary point q_0. We expand the generalized forces in series of $\hat{q}$ leaving only terms of first order of $\hat{q}$. Using the new expressions for the kinetic energy (33) and the generalized forces we get equations in well know form

$$\sum_{j=1}^{3} (a_{ij}\ddot{\hat{q}}_j + b_{ij}\hat{q}_j) = 0, \quad i = 1, 2, 3 \tag{34}$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{35}$$

The coefficients of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$\begin{aligned} a_{11} &= 2l_3m_3(m_1x_1 - M_3x_3) + m_1x_1(M_1x_1 - 2m_3x_3) \\ &\quad + m_3M_3x_3^2 + J_zM + l_3^2m_3M_3 \end{aligned} \tag{36}$$

$$\begin{aligned} a_{12} = a_{21} &= l_3m_3[m_1(2x_1 + l_1) - 2M_3x_3 + l_3M_3] \\ &\quad - m_1m_3x_3(2x_1 + l_1) + m_1M_1x_1(x_1 + l_1) + m_3M_3x_3^2 \end{aligned} \tag{37}$$

$$a_{13} = a_{31} = l_3m_3[m_1x_1 + M_3(l_3 - x_3)] \tag{38}$$

$$\begin{aligned} b_{11} &= (x_1(m_1(6m_3\omega_0^2(x_3 - l_3) - F) + FM) \\ &\quad - 3m_1M_1\omega_0^2x_1^2 + m_3(-Fx_3 - 3M_3\omega_0^2(l_3 - x_3)^2)) \\ &\quad + Fl_3m_3 - \frac{3(J_z - J_x)M\mu}{r^3} \end{aligned} \tag{39}$$

$$\begin{aligned} b_{12} = b_{21} - \tfrac{3\mu J_{21}}{r^3} &= x_1(m_1(-3\omega_0^2(2m_3(l_3 - x_3) \\ &\quad + l_1M_1) - F) + FM) - 3m_1M_1\omega_0^2x_1^2 \\ &\quad + m_3(3\omega_0^2(x_3 - l_3)(M_3(l_3 - x_3) + l_1m_1) - Fx_3) \\ &\quad + Fl_3m_3 - \tfrac{3(J_z - J_x)\mu M}{r^3} \end{aligned} \tag{40}$$

$$b_{13} = b_{31} = l_3m_3(F - 3\omega_0^2(m_1x_1 + M_3(l_3 - x_3))) \tag{41}$$

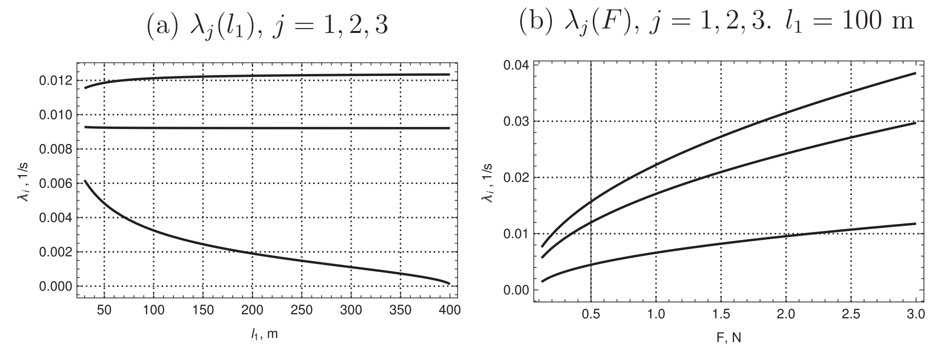

Fig. 5. The natural frequencies of the system as functions of the tether length l_1 for $F = 0.3$ N and as functions of F for $l_1 = 100$ m, $j = 1,2,3$.

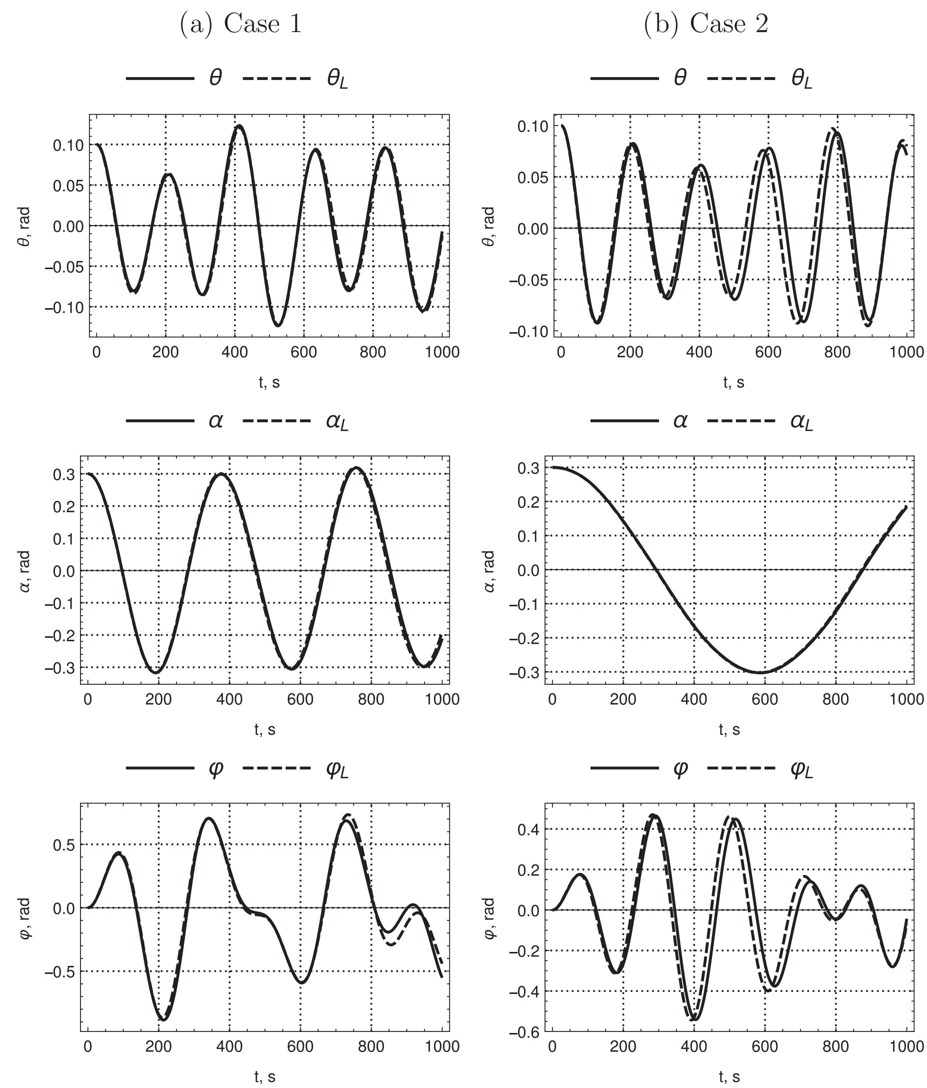

Fig. 6. The solutions of the nonlinear θ, α, φ and linearized $\theta_L, \alpha_L, \varphi_L$ equations for to cases $l_1 = 30$ m (case 1) and $l_1 = 300$ m (case 2).

$$a_{22} = -2l_1m_1(-M_1x_1 + m_3x_3 - l_3m_3) + 2l_3m_3(m_1x_1 - M_3x_3) + m_1x_1(M_1x_1 - 2m_3x_3) + m_3M_3x_3^2 + l_3^2m_3M_3 + l_1^2m_1M_1 \tag{42}$$

$$a_{23} = a_{32} = l_3m_3[m_1(x_1 + l_1) + M_3(l_3 - x_3)] \tag{43}$$

$$b_{22} = l_3m_3(6\omega_0^2(M_3x_3 - m_1(l_1 + x_1)) + F) + F(M_1(x_1 + l_1) - m_3x_3) - 3\omega_0^2(m_1M_1(x_1 + l_1)^2 + m_3M_3x_3^2 - 2m_1m_3x_3(x_1 + l_1)) - 3l_3^2m_3M_3\omega_0^2 \tag{44}$$

$$b_{23} = b_{32} = l_3m_3\{F - 3\omega_0^2[m_1(x_1 + l_1) + M_3(l_3 - x_3)]\} \tag{45}$$

$$a_{33} = l_3^2 m_3 M_3 \tag{46}$$

$$b_{33} = l_3 m_3 (F - 3 l_3 M_3 \omega_0^2) \tag{47}$$

Using the linear Eqs. (34) we can get the natural frequencies of the system. The solutions of the Eqs. (34) have the form

$$q_j = C_j \sin \lambda t, \quad j = 1, 2, 3 \tag{48}$$

Substituting (48) into (34) we get

$$\det(\boldsymbol{A}\lambda^2 - \boldsymbol{B}) = 0 \tag{49}$$

that allows us to find three frequencies $\lambda_1, \lambda_2, \lambda_3$. Fig. 5a shows the frequencies of the system as functions of tether length. Fig. 5b and as functions of the thrust F.

4. Numerical example

Let us compare the solutions of the nonlinear system (19) to the solutions of the linearized system (34) with the following initial conditions

$$\theta_0 = 0.1,\ \alpha_0 = 0.3,\ \varphi_0 = 0,\ \dot\theta_0 = \dot\alpha_0 = \dot\varphi_0 = 0. \tag{50}$$

The parameters of the system are shown in the Table 1.

The solutions are obtained for two cases. Case 1 for $l_1 = 30$ m, $F = 2$N and case 2 for $l_1 = 300$ m, $F = 2$N. The simulation results are shown in Fig. 6. The comparison of results show good accuracy of the approximate linearized model.

5. Conclusion

The motion equations of debris with the fuel residuals during tethered transportation are derived. The stationary solutions of the equations are found. The linearized differential equations with constant coefficients governing the motion of the system near the stationary point are derived. It has been demonstrated that the solutions obtained by means of the linearized system are in good agreement with the solutions of the original nonlinear system of equations. The proposed simplified equations can be readily used in practice to investigate the motion of the tug–debris system with fuel residuals for different parameters of the system.

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Appendix A. The nonlinear equations of motion

$$\begin{aligned}
&(J_2\ddot\theta + J_2\ddot u)M^2 + l_3 m_3\{x_3\big(m_3(m_2 - 2M) + m_3^2 + M^2\big)\\
&+m_1(x_1(m_2 + m_3 - 2M) + m_3x_3) + m_1^2x_1\}\dot\beta^2 s_{\beta-\gamma}\\
&+l_1m_1\{l_3m_3Mc_{\alpha-\gamma}\\
&+[x_1\big(m_1(m_2 + m_3 - 2M) + m_1^2 + M^2\big) - m_3Mx_3]c_{\alpha-\beta}\}\ddot\alpha\\
&+l_3m_3\{l_3\big(m_3(m_1 + m_2 - 2M) + m_3^2 + M^2\big)\\
&-[x_3\big(m_3(m_2 - 2M) + m_3^2 + M^2\big)\\
&+m_1(x_1(m_2 + m_3 - 2M) + m_3x_3) + m_1^2x_1]c_{\beta-\gamma}\}\ddot\gamma\\
&+\{-l_3m_3[x_3(m_3(m_2 - 2M) + m_3^2 + M^2)\\
&+m_1(x_1(m_2 + m_3 - 2M) + m_3x_3) + m_1^2x_1]c_{\beta-\gamma}\\
&+m_1[2m_3x_3x_1(m_2 + m_3 - 2M) + m_3^2x_3^2 + M^2x_1^2]\\
&+m_3x_3^2\big(m_3(m_2 - 2M) + m_3^2 + M^2\big)\\
&+m_1^2x_1(x_1(m_2 + m_3 - 2M) + 2m_3x_3) + m_1^3x_1^2\}\ddot\beta\\
&+2l_3m_3\{x_3[m_3(m_2 - 2M) + m_3^2 + M^2]\\
&+m_1[x_1(m_2 + m_3 - 2M) + m_3x_3] + m_1^2x_1\}\ \omega_0\dot\varphi s_\varphi\\
&+2l_1m_1\{x_1[m_1(m_2 + m_3 - 2M) + m_1^2 + M^2]s_\theta\\
&+Ml_3m_3s_{\theta+\varphi} - m_3Mx_3s_\theta\}\omega_0\dot\alpha \quad =\\
&l_3m_3\{x_3[m_3(m_2 - 2M) + m_3^2 + M^2]\\
&+m_1(x_1(m_2 + m_3 - 2M) + m_3x_3) + m_1^2x_1\}\dot\gamma^2 s_{\beta-\gamma}\\
&+\dot\alpha^2 l_1m_1\{l_3m_3Ms_{\alpha-\gamma} - m_3Mx_3s_{\alpha-\beta}\\
&+x_1[m_1(m_2 + m_3 - 2M) + m_1^2 + M^2]s_{\alpha-\beta}\} + 3J_{21}\mu M^2p^{-3}c_\beta s_\beta\\
&-Fl_3m_3Ms_\gamma - F[x_1(m_1(m_2 + m_3 - 2M) + m_1^2 + M^2)\\
&-m_3Mx_3]s_\beta\\
&\tfrac{3\omega_0}{2}\{l_3^2m_3\big(m_3(m_1 + m_2 - 2M) + m_3^2 + M^2\big)s_{2\gamma}\\
&-2l_3m_3[(x_3(m_1^2x_1 + m_3(m_2 - 2M) + m_3^2 + M^2)\\
&+m_1(x_1(m_2 + m_3 - 2M) + m_3x_3))s_{2\alpha+2\theta+\varphi} - l_1m_1Mc_\gamma s_\alpha]\\
&c_\beta\{l_1m_1s_\alpha\big(x_1\big(m_1(m_2 + m_3 - 2M) + m_1^2 + M^2\big) - m_3Mx_3\big)\\
&+s_\beta\{m_1(2m_3x_3x_1(m_2 + m_3 - 2M) + m_3^2x_3^2 + M^2x_1^2)\\
&+m_3x_3^2(m_3(m_2 - 2M)\\
&+m_3^2 + M^2) + m_1^2x_1(x_1(m_2 + m_3 - 2M) + 2m_3x_3) + m_1^3x_1^2\}\}
\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
&-3(m_1(M^2+m_1^2+m_1(-2M+m_2+m_3))s_{2\alpha}l_1^2\\
&+2m_1(Ml_3m_3s_{2\alpha+\theta+\varphi}\\
&+s_{2\alpha+\theta}((M^2+m_1^2+m_1(-2M+m_2+m_3))x_1-Mm_3x_3))l_1\\
&+l_3^2m_3(M^2+m_3^2+(-2M+m_1+m_2)m_3)s_{2\gamma}\\
&-2l_3m_3s_{2\alpha+2\theta+\varphi}(x_1m_1^2+((-2M+m_2+m_3)x_1+m_3x_3)m_1\\
&+(M^2+m_3^2+(m_2-2M)m_3)x_3)+s_{2\beta}(x_1^2m_1^3\\
&+x_1((-2M+m_2+m_3)x_1+2m_3x_3)m_1^2\\
&+(M^2x_1^2+2m_3(-2M+m_2+m_3)x_3x_1+m_3^2x_3^2)m_1\\
&+m_3(M^2+m_3^2+(m_2-2M)m_3)x_3^2))\omega_0^2\\
&-4\dot{\theta}l_1m_1(Ml_3m_3s_{\theta+\varphi}+s_\theta((M^2+m_1^2\\
&+m_1(-2M+m_2+m_3))x_1-Mm_3x_3))\omega_0\\
&+4\dot{\varphi}l_3m_3(s_\varphi(x_1m_1^2+((-2M+m_2+m_3)x_1+m_3x_3)m_1\\
&+(M^2+m_3^2+(m_2-2M)m_3)x_3)\\
&-Ml_1m_1s_{\theta+\varphi})\omega_0+2F(l_1(M^2+m_1^2+m_1(-2M+m_2+m_3))s_\alpha\\
&+Ml_3m_3s_\gamma+s_\beta((M^2+m_1^2+m_1(m_2+m_3-2M))x_1\\
&-Mm_3x_3))\\
&+2(-l_1m_1(Ml_3m_3s_{\alpha-\gamma}+s_{\alpha-\beta}((M^2+m_1^2\\
&+m_1(-2M+m_2+m_3))x_1-Mm_3x_3))\dot{\alpha}^2\\
&+\ddot{\alpha}l_1m_1(Mc_{\alpha-\gamma}l_3m_3+l_1(M^2+m_1^2+m_1(-2M+m_2+m_3))\\
&+c_{\alpha-\beta}((M^2+m_1^2+m_1(-2M+m_2+m_3))x_1-Mm_3x_3))\\
&-\dot{\gamma}^2l_3m_3(s_{\beta-\gamma}(x_1m_1^2+((-2M+m_2+m_3)x_1+m_3x_3)m_1\\
&+(M^2+m_3^2+(m_2-2M)m_3)x_3)-Ml_1m_1s_{\alpha-\gamma})\\
&+\dot{\beta}^2(l_1m_1s_{\alpha-\beta}((M^2+m_1^2+m_1(-2M+m_2+m_3))x_1-Mm_3x_3)\\
&+l_3m_3s_{\beta-\gamma}(x_1m_1^2+((-2M+m_2+m_3)x_1+m_3x_3)m_1\\
&+(M^2+m_3^2+(m_2-2M)m_3)x_3))\\
&+\ddot{\gamma}l_3m_3(Mc_{\alpha-\gamma}l_1m_1+l_3(M^2+m_3^2+(-2M+m_1+m_2)m_3)\\
&-c_{\beta-\gamma}(x_1m_1^2+((-2M+m_2+m_3)x_1+m_3x_3)m_1\\
&+(M^2+m_3^2+(m_2-2M)m_3)x_3))+\ddot{\beta}(c_\beta m_1M_1x_1(c_\gamma l_3m_3\\
&-c_\beta x_3m_3+M_1(c_\alpha l_1+c_\beta x_1))-c_\beta m_1m_3x_3(c_\gamma l_3m_3\\
&-c_\beta x_3m_3+M_1(c_\alpha l_1+c_\beta x_1))+c_\beta m_2(m_1x_1+m_3x_3)(c_\alpha l_1m_1\\
&-c_\gamma l_3m_3+c_\beta(m_1x_1+m_3x_3))\\
&+m_2s_\beta(m_1x_1+m_3x_3)(l_1m_1s_\alpha-l_3m_3s_\gamma\\
&+s_\beta(m_1x_1+m_3x_3))-m_1m_3s_\beta x_1(M_3(s_\beta x_3-l_3s_\gamma)\\
&-m_1(l_1s_\alpha+s_\beta x_1))\\
&+m_3M_3s_\beta x_3(M_3(s_\beta x_3-l_3s_\gamma)-m_1(l_1s_\alpha+s_\beta x_1))\\
&+c_\beta m_1m_3x_1(m_1(c_\alpha l_1+c_\beta x_1)+M_3(c_\gamma l_3-c_\beta x_3))\\
&-c_\beta m_3M_3x_3(m_1(c_\alpha l_1+c_\beta x_1)+M_3(c_\gamma l_3-c_\beta x_3))\\
&+m_1s_\beta(M_1x_1-m_3x_3)(l_1M_1s_\alpha+l_3m_3s_\gamma\\
&+s_\beta(M_1x_1-m_3x_3))))=0
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&-3\omega_0^2(2l_1m_1Mc_\gamma s_\alpha-2m_1^2x_1c_\gamma s_\beta-2m_2m_3x_3c_\gamma s_\beta\\
&+l_3(m_3(m_1+m_2-2M)+m_3^2+M^2)s_{2\gamma}+2m_1Mx_1s_{2\alpha+2\theta+\varphi}\\
&+2m_3Mx_3s_{2\alpha+2\theta+\varphi}-2m_1Mx_1s_\varphi-2m_3Mx_3s_\varphi\\
&-m_1m_2x_1s_{2\alpha+2\theta+\varphi}-m_1m_3x_1s_{2\alpha+2\theta+\varphi}-m_3^2x_3s_{2\alpha+2\theta+\varphi}\\
&-m_1m_3x_3s_{2\alpha+2\theta+\varphi}+m_1m_2x_1s_\varphi+m_1m_3x_1s_\varphi\\
&+m_3^2x_3s_\varphi+m_1m_3x_3s_\varphi-M^2x_3s_{2\alpha+2\theta+\varphi}+M^2x_3s_\varphi)\\
&+2(l_1m_1M\ddot{\alpha}c_{\alpha-\gamma}-\ddot{\beta}c_{\beta-\gamma}(x_3(m_3(m_2-2M)+m_3^2+M^2)\\
&+m_1(x_1(m_2+m_3-2M)+m_3x_3)+m_1^2x_1)\\
&+l_3\ddot{\gamma}(m_3(m_1+m_2-2M)+m_3^2+M^2)\\
&+\dot{\alpha}^2l_1m_1(-M)s_{\alpha-\gamma}+\dot{\beta}^2(x_3(m_3(m_2-2M)+m_3^2+M^2)\\
&+m_1(x_1(m_2+m_3-2M)+m_3x_3)+m_1^2x_1)s_{\beta-\gamma})+2FMs_\gamma\\
&-4\dot{\alpha}\omega_0(s_\varphi(m_1(x_1(m_2+m_3-2M)+m_3x_3)+M_3^2x_3)\\
&-l_1m_1Ms_{\theta+\varphi})-4\dot{\theta}\omega_0s_\varphi(m_1(x_1(m_2+m_3-2M)+m_3x_3)+M_3^2x_3)\\
&-4\dot{\beta}\omega_0(m_1^2x_1+m_2m_3x_3)s_\varphi=0
\end{aligned}
\tag{A.3}
$$

where $\gamma = \alpha + \theta + \varphi, \beta = \alpha + \theta$.

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