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Transient Modes of Spacecraft Angular Motion on the Upper Section of the Reentry Trajectory

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Abstract—The motion of an uncontrolled spacecraft around its center of mass is considered, the restoring aerodynamic moment of the spacecraft being described by an odd Fourier series in the angle of attack with the two first harmonics. The evolution of phase trajectories is studied on the basis of analysis of an action integral, for which the analytical formulas are obtained in full elliptic integrals or elementary functions. The moments of transition between various phase plane regions and boundary conditions for kinematic parameters of motion are determined. For the cases of motion, when, intersecting the separatrix, the phase point may fall into various oscillation regions, the formulas for determining the possibility of capture into any region are found.

1. FORMULATION OF THE PROBLEM

The motion of an axisymmetric spacecraft around its center of mass on the upper section of the atmospheric reentry trajectory is considered. In this case, the variation of the velocity of the center of mass, the trajectory inclination angle, and the aerodynamic damping may be neglected. The cases are investigated in which the character of motion changes during the reentry process: the rotational motion transfers into an oscillatory one, and the oscillatory motion transfers "by jumping" into the oscillatory motion with other amplitude characteristics. The case of planar angular motion of spacecraft with a sinusoidal angle of attack dependence of the restoring moment is considered in [1]. This paper considers both planar and spatial motion around a spacecraft's center of mass with the angle of attack dependence of the restoring moment having a form of a biharmonical series. Such an angle of attack dependence of the restoring moment is typical for uncontrolled reentry vehicles of segmentally-conical, blunted conical, and other shapes (*Soyuz*, *Mars*, *Apollo*, *Viking*, and other reentry modules). The presence of the second harmonic in moment characteristics causes the possibility of appearance of an additional equilibrium position of a spacecraft in the angle of attack, i.e., an additional singular point on a phase portrait of the system, which causes the appearance of a series of new cases of transient modes.

The motion of a spacecraft with the biharmonic moment characteristics around the center of mass under the aforementioned assumptions is described by the system with slowly varying parameters of type [2]

$$\begin{gathered}\ddot{\alpha} + F(\alpha) = 0,\\ F(\alpha) = (G - R\cos\alpha)(R - G\cos\alpha)/\sin^3\alpha\\ +a\sin\alpha + b\sin 2\alpha = 0,\\ a = a(z),\ b = b(z),\end{gathered} \quad (1)$$

where α is the angle of attack; $R = \text{const}$, $G = \text{const}$ are projections of a kinetic moment vector on the longitudinal axis of a spacecraft, normalized with respect to the transversal moment of inertia; $a(z)$, $b(z)$ are moment characteristic coefficients; z is a slowly varying parameter.

The energy integral of system (1) for constants a and b has the form of

$$\dot{\alpha}^2/2 + W(\alpha) = h, \quad (2)$$

where

$$W(\alpha) = \int F(\alpha)d\alpha = 0.5(G^2 + R^2 - 2GR\cos\alpha)/\sin^2\alpha - a\cos\alpha - b\cos^2\alpha.$$

The type of the system's motion is determined by the relation between quantities a, b, R, G, and h. In the planar case of motion ($R = G = 0$), three types of phase portraits take place.

(1) $|a| \geq 2|b|$. The phase portrait is analogous to an oscillatory system of the pendulum type and is depicted for $a > 0$ in Fig. 1 (for $a < 0$, the phase picture is shifted by value π along the α-axis).

(2) $b > 0.5|a|$, $b > 0$. Some additional singular points of saddle type appear on the phase portrait. These points correspond to the values of the angle of attack $\alpha_* = \pm\arccos(-0.5a/b) + 2n\pi$ ($n = 0, \pm1, \pm2, \ldots$), and three regions of motion take place: a rotational and two oscillatory ones (Fig. 2).

(3) $|b| > 0.5|a|$, $b < 0$. The phase portrait for the case $a > 0$ is shown in Fig. 3 (for $a < 0$, the phase picture is shifted by value π along the α-axis). Here some singular points of center type correspond to the values of the angle of attack $\alpha_* = \pm\arccos(-0.5a/b) + 2n\pi$ ($n = 0, \pm1, \pm2, \ldots$), and four regions of motion take place, a rotational and three oscillatory ones (Fig. 3).

In the spatial case of motion, the presence of a gyroscopic term in equation (1) stipulates the exclusively oscillatory character of spacecraft motion. The presence of the second harmonic in the moment characteristics causes a possibility of appearance of some singular point of saddle type on the phase portrait of a system. In this case, there are three oscillation regions (Fig. 4).

In connection with the change of a and b coefficients during the motion, the evolution of phase trajectories takes place. As a result, these trajectories can intersect separatrices and fall into various phase portrait regions, which is followed by qualitative changes in the motion character. Figure 5 shows one of the possible versions of an angle of attack variation in the case of a spacecraft's spatial motion around the center of mass during descent.

Coefficients a and b, whose variability is associated with the atmospheric density variation during the descent, may be represented in the form [2]

$$a = c_a z, \quad b = c_b z, \tag{3}$$

$$c_a = -m_a S l \rho_{(1)} V_0^2/(2I), \quad c_b = -m_b S l \rho_{(1)} V_0^2/(2I),$$

$$z = \exp(\beta t), \quad \beta = \lambda V_0|\sin\theta_0|,$$

where m_a, m_b are constant coefficients, S is the characteristic area, l is the characteristic dimension, I is the transversal moment of inertia of a spacecraft, V_0 is the flight velocity, θ_0 is the trajectory inclination angle, $\rho_{(1)}$ is the atmospheric density at the "atmosphere boundary" at time $t = t_1$, and λ is the logarithmic density gradient in height.

To describe the motion of a system with slowly varying parameters (1), we shall use the integral of action written in the form of

$$I = \int_{\alpha_{min}}^{\alpha_{max}} \dot{\alpha}\, d\alpha, \tag{4}$$

where α_{min}, α_{max} are amplitude values of the angle of attack (in a planar rotation, $\alpha_{min} = -\pi$, $\alpha_{max} = \pi$; in planar oscillations, $\alpha_{min} = -\alpha_{max}$); $\dot{\alpha}$ is determined from the energy integral (2).

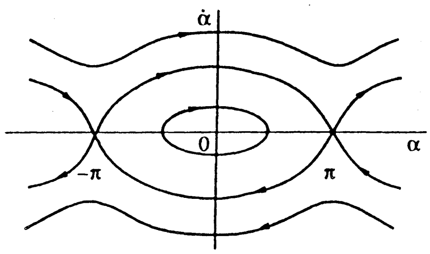

Fig. 1. Phase portrait of the planar motion: $|a| \geq 2|b|$.

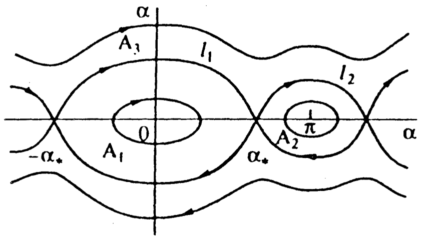

Fig. 2. Phase portrait of the planar motion: $b > 0.5|a|$, $b > 0$.

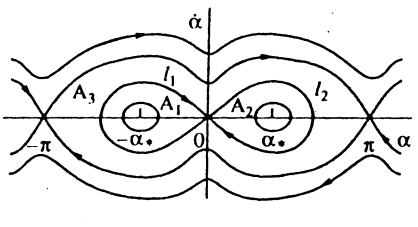

Fig. 3. Phase portrait of the planar motion: $|b| > 0.5|a|$, $b < 0$.

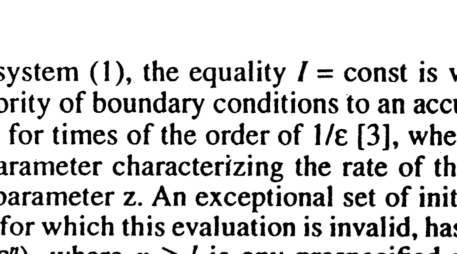

Fig. 4. Phase portrait of the spatial motion.

For system (1), the equality I = const is valid for the majority of boundary conditions to an accuracy of $O(\varepsilon\ln\varepsilon)$ for times of the order of $1/\varepsilon$ [3], where ε is a small parameter characterizing the rate of the variation of parameter z. An exceptional set of initial conditions, for which this evaluation is invalid, has a measure $O(\varepsilon^n)$, where $n \geq l$ is any prespecified number. The motion modes corresponding to the given initial conditions are called the modes of spacecraft hovering in the unstable equilibrium vicinity. These modes were thoroughly investigated in [2] and are not considered here.

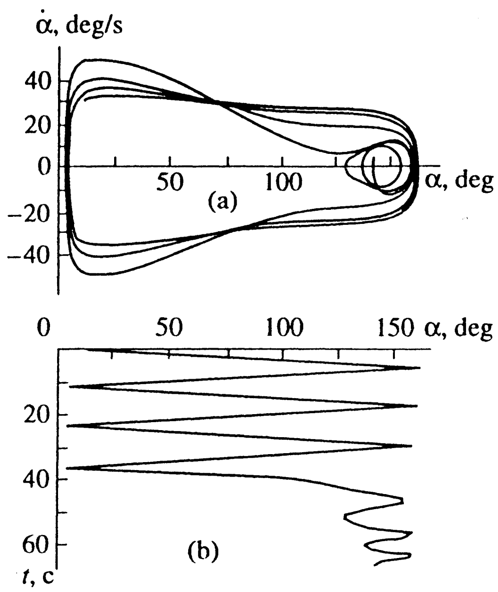

Fig. 5. Character of spatial motion variation during the reentry: (a) phase trajectories; (b) variation of the angle of attack.

In this work, the analysis of transient modes of motion is based on analytical expressions for the action integral (4). The time moments corresponding to the transitions between various phase portrait regions are determined from the equality of the action integral expression calculated along the separatrices to the action integral value calculated from the initial conditions of motion.

The value of the angle of attack at the boundary of transition from one type of motion to another depends, in the general case, upon the distribution of initial angles of attack and angular velocities at the atmosphere boundary, as well as upon the rate of change of a and b coefficients [2]. It is assumed that, for the time of motion before reaching the transition boundary, the vehicle makes several turns or oscillations, so that the angle of attack may be supposed to be a random variable uniformly distributed within the range under consideration. In this case, the angular velocity value $\dot{\alpha}$ is determined from energy integral (2).

In cases when, at intersecting separatrices, the phase point may fall into various oscillation regions, the problem of choosing a motion continuation region arises. Let separatrices l_1 and l_2 separate the inner regions of motion A_1, A_2 from the outer one, A_3 (see Figs. 2, 3, 4). For choosing the motion continuation region A_1 or A_2, the probability P_i, $i = 1, 2$, of capture into each is used. In accordance with [4], this probability is defined as the fraction of a phase volume of a small neighborhood around the initial point of motion, which is "captured" into the region under consideration in the limit when a small parameter $\varepsilon \longrightarrow 0$ and the dimension of neighborhood $\delta \longrightarrow 0$, $\varepsilon \ll \delta$ (the limit is first taken over ε and then, over δ, where $P_1 + P_2 = 1$). The ratio of probabilities is calculated by formulas

$$\frac{P_1}{P_2} = \frac{\theta_1}{\theta_2}, \tag{5}$$

$$\theta_i = -\oint_{l_i} \frac{\partial E}{\partial z} f_z dt \quad (i = 1, 2), \tag{6}$$

$$E = H(p, q, z) - H(0, q_*, z),$$

where $H = p^2/2 + W(q, z)$ is the Hamiltonian; $p = \dot{\alpha}$, $q = \alpha$ are canonical variables; $p = 0$, $q_* = \alpha_*$ are coordinates of a singular saddle-type point on the phase portrait ($E = 0$ at a singular saddle-type point and on separatrices, $E > 0$ in A_3, $E < 0$ in $A_{1,2}$); $f_z = \dot{z} = \beta z$. Integrals (6) are calculated along the separatrices l_1 and l_2, parametrized by time t of undisturbed motion along these separatrices.

One should note that, since we consider the upper section of the reentry trajectory for which $f_z > 0$, the quantities θ_i will also be positive; therefore, a single passage through the phase point separatrix from the outer region into the inner one takes place [4].

Therefore, with the known initial conditions of motion at the atmosphere boundary, one may trace the phase trajectory evolution, find the times of transition into each characteristic region of a phase portrait, and determine the boundary conditions of motion for this region.

2. THE PLANAR MOTION

The analysis of transient modes will be carried out for three aforementioned cases of planar motion, the phase portraits of which are shown in Figs. 1–3, and also for an additional special case, when $a = 0$, $b \neq 0$.

(1) $|a| \geq 2|b|$. The phase portrait of a system is depicted in Fig. 1 (for definiteness, we shall consider the case of $a > 0$, since this condition can always be achieved in equation (1) by a proper choice of the reference system). The rotational motion of a spacecraft around the center of mass is governed by the condition $h > a - b$. Let us find the analytical expression for the action integral in the rotational motion. We introduce a new variable $u = \cos\alpha$; then the integral (4) for $R = G = 0$ takes the form

$$I = -2\int_1^{-1} \frac{\sqrt{f(u)}}{1 - u^2} du, \tag{7}$$

where

$$f(u) = 2(1 - u^2)(h + au + bu^2). \tag{8}$$

The result of integration of expression (7) depends on the sign of a highest-order coefficient of polynomial $f(u)$ and on the types of its roots. Two of the four roots of polynomial (8) are unambiguously real and correspond to the values of ±1. The two remaining roots may be either real or complex-conjugated, depending on the relation between quantities h, a, b. The following expressions for the action integral take place:

For $b > 0$, $h > a^2/(4b)$ (two real and two complex-conjugated roots),

$$I = \eta\{hK(k) - |a|(1 + 1/n)^{1/2}[K(k) - \Pi(n, k)] + b[K(k) + (E(k) - (1+n)\Pi(n, k))/(k^2 + n)]\}, \quad (9)$$

where $K(k)$, $E(k)$, $\Pi(n, k)$ are full elliptic integrals of the first, second, and third kind; $k = [0.5 - (h - b)/(pr)]^{1/2}$ is the modulus of elliptic integrals, $n = (r - p)^2/(4pr)$, $\eta = 8/(pr)^{1/2}$; $p, r = [2(h \pm a + b)]^{1/2}$.

For $b > 0$, $h = a^2/(4b)$ (all roots are real, two roots being equal to each other, the modulus of elliptic integrals $k = 0$),

$$I = 2\pi a/\sqrt{2b}. \quad (10)$$

For $b > 0$, $[a^2/(4b)] \geq h > [a - b]$ and $b < 0$, $h > a - b$ (all roots are real),

$$I = \eta\left\{hK(k) + a\left[\left(1 + \frac{2}{n}\right)K(k) - \left(2 + \frac{2}{n}\right)\Pi(n, k)\right]\right.$$
$$+ b\left[\frac{1 + (1+n)^2}{n^2}K(k) + \frac{2(1+n)}{n(k^2+n)}E(k) \quad (11)\right.$$
$$\left.\left.+ \frac{2(1+n)^2}{n^2}\left(\frac{2k^2+n}{k^2+n} + \frac{1 - 2(2+n)}{1+n}\right)\Pi(n, k)\right]\right\},$$

where $k = [2(p - r)/((p - 1)(1 + r))]^{1/2}$, $n = 2/(p - 1)$, $\eta = 8/[2b(p - 1)(1 + r)]^{1/2}$; $p, r = -a/(2b) \pm \operatorname{sgn}(b) \times [(a/(2b))^2 - h/b]^{1/2}$.

For $b = 0$, we have a particular case of motion with a sinusoidal moment characteristic, and the action integral is determined by the formula

$$I = 4\sqrt{2(h + a)}E(k), \quad (12)$$

where $k = [2a/(h + a)]^{1/2}$.

The transition boundary of rotation into oscillations is described by the $h = a - b$ condition (all roots of polynomial (8) are real, two roots are equal to each other, the modulus of elliptic integrals $k = 1$). At the transition boundary, at time $t = t_2$, the following expressions for an action integral, which is taken along the separatrices, are valid:

$$I_{(2)} = 4\sqrt{2b_{(2)}}[\sqrt{u_* - 1} + u_*\arctan(\sqrt{1/(u_* - 1)})], \text{ for } b > 0, \quad (13)$$

$$I_{(2)} = 4\sqrt{-2b_{(2)}}[\sqrt{u_* + 1} + u_*\ln((1 + \sqrt{u_* + 1})/\sqrt{u_*})], \text{ for } b < 0, \quad (14)$$

$$I_{(2)} = 8\sqrt{a_{(2)}}, \text{ for } b = 0, \quad (15)$$

where $u_* = |a/(2b)| = |m_a/(2m_b)|$.

Using formulas (13)–(15) and taking into account the constancy of an action integral, $I_{(2)} = I_{(1)}$, one can determine the value of coefficients b and a at time t_2 in terms of initial conditions of rotational motion:

$$b_{(2)} = \{I_{(1)}/[\sqrt{u_* - 1} + u_*\arctan(\sqrt{1/(u_* - 1)})]\}^2/32 \text{ for } b > 0, \quad (16)$$

$$b_{(2)} = -\{I_{(1)}/[\sqrt{u_* + 1} + u_*\ln((1 + \sqrt{u_* + 1})/\sqrt{u_*})]\}^2/32 \text{ for } b < 0, \quad (17)$$

$$a_{(2)} = I_{(1)}^2/64 \text{ for } b = 0. \quad (18)$$

Here the action integral $I_{(1)}$ is calculated from the initial conditions of motion by using one of formulas (9)–(12). In the case when, at the atmosphere boundary, $a_{(1)}$ and $b_{(1)}$ coefficients are essentially small as compared to the angular velocity $\dot{\alpha}_{(1)}$, the action integral $I_{(1)}$ can be determined by the well-known formula [2]

$$I_{(1)} = 2\pi\dot{\alpha}_{(1)}. \quad (19)$$

The time corresponding to the moment of transition of rotation into oscillations is determined from expressions (3) by using formulas (16)–(18),

$$t_2 = (\ln[b_{(2)}/c_b]/\beta), \quad (20)$$

or

$$t_2 = \ln[a_{(2)}/c_a]/\beta. \quad (21)$$

One should note that, according to equalities (3), coefficients a and b in the case $b \neq 0$ are related by the equality $a = b(m_a/m_b)$.

Let us determine the boundary conditions for the angle of attack and the angular velocity. We suppose that, for the time of rotational motion $t_2 - t_1$, the vehicle performs several turns. Then the angle of attack $\alpha = \alpha_{(2)}$ corresponding to the time $t = t_2$ may be considered to be a random variable, uniformly distributed in the range from $-\pi$ to π (to an accuracy of period 2π). For the angular velocity $\dot{\alpha}_{(2)}$, taking into account the energy integral (2) and equality $h_{(2)} = a_{(2)} - b_{(2)}$, we have the formula

$$\dot{\alpha}_{(2)} = \operatorname{sgn}(\dot{\alpha}_{(1)})[2a_{(2)}(1 + \cos\alpha_{(2)}) - 2b_{(2)}\sin^2\alpha_{(2)}]^{1/2}. \quad (22)$$

(2) $b > 0.5|a|$, $b > 0$. The phase portrait is shown in Fig. 2. Coefficient a may take both positive and negative values. The $h > a^2/(4b)$ condition corresponds to the rotational motion of a spacecraft around the center of mass. The action integral in the rotational motion is determined by expression (9), since, in this case, two roots of polynomial (8) are real and two others are complex-conjugated.

The $h = a^2/(4b)$ condition corresponds to the boundary of transition of rotation into oscillations. At the transition boundary at time $t = t_2$, the action integral, taken along the separatrices, has the form

$$I_{(2)} = 4\sqrt{2b_{(2)}}[\sin\alpha_* + (0.5\pi - \alpha_*)\cos\alpha_*], \quad (23)$$

where $\alpha_* = \arccos(-0.5a/b)$.

Taking into account that $I_{(2)} = I_{(1)}$, we determine from equality (23) the value of coefficient b at time t_2 through initial conditions of the rotational motion

$$b_{(2)} = [I_{(1)}/(\sin\alpha_* + (0.5\pi - \alpha_*)\cos\alpha_*)]^2/32. \quad (24)$$

The time corresponding to the moment of transition of rotation into oscillations is determined by formula (20).

Since at $t > t_2$ the vehicle can oscillate with respect to one of two stable positions of equilibrium in the angle of attack $\alpha = 0$ or $\alpha = \pi$, let us determine the probability of falling into these oscillation regions. We denote by P_1 the probability of falling into the vicinity of the angle of attack value $\alpha = 0$; by P_2, the probability of falling into the vicinity of $\alpha = \pi$, where $P_1 + P_2 = 1$. After appropriate calculations by formulas (5) and (6), we have

$$\frac{P_1}{P_2} = \frac{1 - \alpha_*\cot\alpha_*}{1 + (\pi - \alpha_*)\cot\alpha_*}. \quad (25)$$

As seen, the value of the probability of the vehicle falling into any oscillation region is determined only by the value of an unstable position of equilibrium in the angle of attack $\alpha = \alpha_*$.

Let us determine the boundary conditions for the angle of attack and the angular velocity. The angle of attack $\alpha = \alpha_{(2)}$ corresponding to the time $t = t_2$ will be assumed to be a random variable uniformly distributed in the range from $-\alpha_*$ to α_* or in the range from α_* to $2\pi - \alpha_*$. The probability of falling into these ranges is determined by formula (25). Taking into account the energy integral (2) and the equality $h = a^2/(4b)$, we have for the angular velocity $\dot{\alpha}_{(2)}$ the expression

$$\dot{\alpha}_{(2)} = \operatorname{sgn}(\dot{\alpha}_{(1)})[2a_{(2)}\cos\alpha_{(2)}, + 2b_{(2)}\cos^2\alpha_{(2)} + a_{(2)}^2/(2b_{(2)})]^{1/2}, \text{ or} \quad (26)$$

$$\dot{\alpha}_{(2)} = \operatorname{sgn}(\dot{\alpha}_{(1)})\sqrt{2b_{(2)}}|\cos\alpha_{(2)} - \cos\alpha_*|,$$

where $\alpha_* = \arccos(-0.5a/b)$.

(3) $|b| > 0.5|a|$, $b < 0$. The phase portrait of a system is depicted in Fig. 1 (for definiteness, we consider the case of $a > 0$, since this condition can always be achieved in equation (1) by a proper choice of the reference system). The rotational motion of the spacecraft around the center of mass corresponds to the condition $h > a - b$. The action integral in the rotational motion is determined by expression (11), since in this case all roots of polynomial (8) are real.

The $h = a - b$ condition corresponds to the boundary of transition of rotation into oscillations with respect to the unstable position of equilibrium $\alpha = 0$. At the transition boundary at time $t = t_2$, the action integral, taken along the separatrices, is determined by expression (14), where, in this case, $u_* = \cos\alpha_* = -0.5a/b$. The value of coefficient b at time t_2 is determined in terms of the initial conditions of rotational motion by equality (17). The time corresponding to the transition of rotation into oscillations is calculated by formula (20).

At $a - b > h > -a - b$, the vehicle oscillates with respect to the unstable equilibrium position $\alpha = 0$. Let us find the analytic expression for the action integral in the oscillatory motion. In this case, all roots of polynomial (8) are real, and one of them corresponds to amplitude values of the angle of attack: $u_m = \cos\alpha_{max} = \cos\alpha_{min} = \cos\alpha_m$, and it is the upper limit of integral (7). As a result of integration, we have

$$I = \eta\Big\{hK(k) - a[K(k) - 2(1+n)\Pi(n,k)] - b\Big[(1+2n)K(k) - \frac{2n(1+n)}{(k^2+n)}E(k) - \frac{2(1+n)(k^2+2k^2n+n^2)}{k^2+n}\Pi(n,k)\Big]\Big\}, \quad (27)$$

where $k = [0.5(1+u_l)(1-u_m)/(u_l-u_m)]^{1/2}$; $n = 0.5(u_m - 1)$, $\eta = 4/[b(u_m - u_l)]^{1/2}$; $u_l, u_m = -a/(2b) \pm [(a/2b)^2 - h/b]^{1/2}$.

One should note that expression (27) is a function of parameters a, b and $\cos\alpha_m$ ($h = -a\cos\alpha_m - b\cos^2\alpha_m$) and represents an implicit specification of the amplitude of oscillations α_m in terms of a, b and the initial value of the action integral.

The $h = -a - b$ condition corresponds to the boundary of transition from the region of oscillations with respect to an unstable equilibrium position $\alpha = 0$ into one of two regions of oscillations with respect to stable equilibrium positions $\alpha_* = \pm\arccos(-0.5a/b)$. At the transition boundary at time $t = t_3$, the expression for an action integral takes the form

$$I_{(3)} = 4\sqrt{-2b_{(3)}}[\sqrt{1-\cos\alpha_*} - \cos\alpha_* \ln((1+\sqrt{1-\cos\alpha_*})/\sqrt{\cos\alpha_*})]. \quad (28)$$

Taking into account that $I_{(3)} = I_{(1)}$, we determine from equality (28) the value of coefficient b at time t_3 in terms of the initial conditions of motion

$$b_{(3)} = \{I_{(1)}/[\sqrt{1-\cos\alpha_*} - \cos\alpha_* \ln((1+\sqrt{1-\cos\alpha_*})/\sqrt{\cos\alpha_*})]\}^2/32. \quad (29)$$

The time corresponding to the given transition will be determined from expressions (3) by using (29),

$$t_3 = \ln[b_{(3)}/c_b]/\beta. \quad (30)$$

The amplitude of oscillations at time t_3 is found from the energy integral (2), taking into account that $h_{(3)} = -a_{(3)} - b_{(3)} = W(\alpha_{m_{(3)}})$,

$$\alpha_{m_{(3)}} = \arccos[2\cos\alpha_* - 1]. \quad (31)$$

Now we determine the boundary conditions for the angle of attack and the angular velocity. The angle of attack $\alpha = \alpha_{(3)}$ corresponding to time $t = t_3$ is considered to be a random variable uniformly distributed in the range from $-\alpha_{m_{(3)}}$ to 0, or in the range from 0 to $\alpha_{m_{(3)}}$. The falling into mentioned ranges is equally probable, since the oscillation regions are equal and symmetrical with respect to a singular saddle point $\alpha = 0$, and we have in formula [5] $\theta_1 = \theta_2$. Taking into account the energy integral (2) and the equality $h_{(3)} = -a_{(3)} - b_{(3)}$, we have for the angular velocity $\dot{\alpha}_{(3)}$ the expression

$$\dot{\alpha}_{(3)} = \pm[2a_{(3)}(\cos\alpha_{(3)} - 1) - 2b_{(3)}\sin^2\alpha_{(3)}]^{1/2}, \quad (32)$$

or

$$\dot{\alpha}_{(3)} = \pm\sqrt{-2b_{(3)}}[2\cos\alpha_*(\cos\alpha_{(3)} - 1) + \sin^2\alpha_{(3)}]^{1/2}.$$

(4) $a = 0, b \neq 0$. In this case, for $b > 0$, the phase portrait of a system has a form shown in Fig. 2, but in this case, the oscillation regions with respect to stable positions of equilibrium in the angle of attack $\alpha = 0$ and $\alpha = \pi$ are equal and symmetrical with respect to an unstable equilibrium position $\alpha = \pi/2$. For $b < 0$, the phase picture is shifted by the $\pi/2$ value along the α-axis.

In the case under consideration, the spacecraft motion is described by the equation

$$\ddot{\alpha} + b\sin 2\alpha = 0. \quad (33)$$

For definiteness, we shall consider the case $b > 0$, since this condition can always be achieved in equation (33) by a proper choice of the reference system.

The energy integral of equation (33) for constant b has the form

$$\dot{\alpha}^2/2 - b\cos^2\alpha = h. \quad (34)$$

The $h > 0$ condition corresponds to the rotational motion of a spacecraft with respect to the center of mass. We write the expression for an action integral in the explicit form. Substituting relation (34) into formula (4) and integrating the result, we obtain

$$I = 4\sqrt{2(h+b)}E(k), \quad (35)$$

where $k = [b/(h+b)]^{1/2}$.

The $h = 0$ ($k = 1$) condition corresponds to the boundary of transition of rotation into oscillations. At the transition boundary at time $t = t_2$, the expression for an action integral takes the form

$$I_{(2)} = 4\sqrt{2b_{(2)}}. \quad (36)$$

Taking into account that $I_{(2)} = I_{(1)}$, we determine from equality (36) the value of coefficient b at time t_2 in terms of the initial conditions of rotational motion,

$$b_{(2)} = I_{(1)}^2/32, \quad (37)$$

or, using formulas (34), (35), we have

$$b_{(2)} = [\dot{\alpha}_{(1)}^2/2 + b_{(1)}(1 - \cos^2\alpha_{(1)})][E(k_{(1)})]^2. \quad (38)$$

The time corresponding to the moment of transition of rotation into oscillations is calculated by formula (20).

Now we determine the boundary conditions for the angle of attack and the angular velocity. The angle of attack $\alpha = \alpha_{(2)}$ corresponding to time $t = t_2$ is considered to be a random variable uniformly distributed in the range from $-\pi/2$ to $\pi/2$ or in the range from $\pi/2$ to $3\pi/2$. The falling into mentioned ranges is equally probable in accordance with formula (25), since, in this case, $\alpha_* = \pi/2$. Taking into account (2) and the equality $h = 0$, we have for angular velocity

$$\dot{\alpha}_{(2)} = \mathrm{sgn}(\dot{\alpha}_{(1)})\sqrt{2b_{(2)}}\cos\alpha_{(2)}.$$

3. THE SPATIAL MOTION

The phase portrait for equation (1) in the case of a spatial motion is determined by the relation between quantities h, a, b, R, and G. The qualitative analysis of equation (1) shows that, if, inside the interval for the angle of attack $(0, \pi)$, the saddle point is absent in the planar case of $R = G = 0$, then it is also absent in the case of spatial motion irrespective of quantities R and G. On the other hand, if, for $R = G = 0$, the saddle point actually takes place (the case of $b > 0.5|a|$, $b > 0$), then its absence can be provided only by choosing sufficiently high (in magnitude) and finite R and G values. We analyze the case when the saddle point exists; then the phase portrait of equation (1) has the form shown in Fig. 4.

Let us find the analytical expression for an action integral in the case when the spacecraft is oscillating in the outer region A_3 (Fig. 4). We introduce the variable $u = \cos\alpha$; then the integral (4) takes the form

$$I = -\int_{u_1}^{u_2} \frac{\sqrt{f(u)}}{1-u^2}du, \quad (39)$$

where

$$f(u) = -2bu^4 - 2au^3 + 2(b-h)u^2 + 2(a+GR)u + (2h - G^2 - R^2), \quad (40)$$
$$u_1 = \cos\alpha_{\min},\ u_2 = \cos\alpha_{\max}.$$

One should note that, in the case of motion inside the outer oscillation region, the polynomial (40) also has—along with the roots corresponding to amplitude values of the angle of attack $u_1 = \cos\alpha_{\min}$, $u_2 = \cos\alpha_{\max}$—two complex-conjugated roots: $u_{3,4} = u_{34} \pm iw$, and $u_2 < u_{34} < u_1$. Substituting polynomial (40) into the formula for action integral (39) and integrating the result, one obtains

$$I = \eta\Bigg\{hK(k) + a[\lambda K(k) + \nu(1+n)\Pi(n,k)] + b\Big[(\lambda^2 - \nu^2(1+n))K(k) + \Big(\frac{\nu^2(1+n)n}{k^2+n}\Big)E(k) + (1+n)\Big(\frac{\nu^2(1+n)(n+2k^2)}{k^2+n} + 2\lambda\nu\Big)\Pi(n,k)\Big] - \sum_{i=1}^{2} 0.5d_i[\lambda_i K(k) + \nu_i(1+n_i)\Pi(n_i,k)]\Bigg\}, \quad (41)$$

where $k = [0.5(1 - \zeta/\vartheta)]^{1/2}$, $n = (\xi - 1)^2/(4\xi)$, $\eta = 4/\sqrt{2b\vartheta}$, $\lambda = (u_1\xi - u_2)/(\xi - 1)$, $\nu = 2\xi(u_2 - u_1)/(\xi^2 - 1)$, $\zeta = (u_1 - u_{34})(u_2 - u_{34}) + w^2$, $\vartheta = [(u_1 - u_{34})^2 + w^2]^{1/2}[(u_2 - u_{34})^2 + w^2]^{1/2}$, $\xi = [(u_1 - u_{34})^2 + w^2]^{-1/2}[(u_2 - u_{34})^2 + w^2]^{1/2}$, $d_{1,2} = 0.5(G \mp R)^2$, $n_{1,2} = (\xi - 1 \pm u_2 \mp \xi u_1)^2/[4\xi(1 \mp u_1 \mp u_2 + u_1u_2)]$, $\lambda_{1,2} = (\xi - 1)/(\xi - 1 \pm u_2 \mp \xi u_1)$, $\nu_{1,2} = (1+\xi)/(1 + \xi \mp u_2 \mp \xi u_1) - \lambda_{1,2}$.

Expression (41) is a function of parameters a, b and $\cos\alpha_{\max} = u_2$, since $h = W(\alpha_{\max})$, and represents an implicit specification of a maximum value of the angle of attack $\alpha_{\max}$ in terms of a, b and the initial value of the action integral. In this case, the minimum value of the angle of attack is $\alpha_{\min} = \arccos u_1$.

The time of transition from an outer oscillation region into one of inner oscillation regions, which are separated by a separatrix, is defined by the time of transition of complex-conjugated roots $u_{3,4} = u_{34} \pm iw$ into real ones $u_{3,4} = u_{34} = u_* = \cos\alpha_*$, $w = 0$; in this case, the modulus of elliptic integrals $k = 1$. Now we obtain the formula for an action integral at time $t = t_2$ corresponding to the mentioned transition. In this case, polynomial (40) has the form

$$f(u) = -2b(u - u_1)(u - u_2)(u - u_*)^2,$$

and integral (39) is calculated in terms of elementary functions as

$$I_{(2)} = \sqrt{2b_{(2)}} \times\Bigg\{2\sqrt{(u_1 - u_*)(u_* - u_2)} - \sum_{i=1}^{3} c_i \arcsin\delta_i\Bigg\}, \quad (42)$$

where $c_1 = u_1 + u_2 + 2u_*$, $c_{2,3} = (1 \mp u_*)\sqrt{(u_1 \mp 1)(u_2 \mp 1)}$, $\delta_1 = (u_1 + u_2 - 2u_*)/|u_2 - u_1|$, $\delta_{2,3} = [(u_2 \mp 1)(u_* - u_1) + (u_1 \mp 1)(u_* - u_2)]/|(u_2 - u_1)(u_* \mp 1)|$.

Taking into account the constancy of an action integral $I_{(2)} = I_{(1)}$, we rewrite expression (42) in the form

$$b_{(2)} = \Bigg\{I_{(1)}\Big/\Bigg[2\sqrt{(u_1 - u_*)(u_* - u_2)} - \sum_{i=1}^{3} c_i \arcsin\delta_i\Bigg]\Bigg\}^2\Big/2. \quad (43)$$

Coefficient $b_{(2)}$ is determined by a simultaneous solution of equation (43) with the determination of roots of polynomial (40). Quantity h is determined from the condition of transition of complex-conjugated roots $u_{3,4} = u_{34} \pm iw$ into real ones $u_{3,4} = u_{34} = u_*$, $w = 0$. The time corresponding to the moment of transition under consideration is calculated by formula (20).

Since the vehicle at $t > t_2$ may continue its motion in one of two inner oscillation regions A_1 or A_2 (Fig. 4), we determine the probability of capturing into these oscillation regions. Using formulas (5) and (6), one obtains

$$\frac{P_1}{P_2} = \Big(\sqrt{(u_1 - u_*)(u_* - u_2)} + (0.5c_1 + m_a/m_b) \times(0.5\pi + \arcsin\delta_1)\Big)\Big/\Big(\sqrt{(u_1 - u_*)(u_* - u_2)} - (0.5c_1 + m_a/m_b)(0.5\pi - \arcsin\delta_1)\Big), \quad (44)$$

with $P_1 + P_2 = 1$.

Now we determine the boundary conditions for the angle of attack and the angular velocity. The angle of attack $\alpha = \alpha_{(2)}$ corresponding to time $t = t_2$ is supposed to be a random variable uniformly distributed in the range from $\alpha_{\min_{(2)}}$ to α_* or in the range from α_* to $\alpha_{\max_{(2)}}$. The probability of falling into the mentioned ranges is determined by formula (44). The angular velocity $\dot{\alpha}_{(2)}$ is determined from the energy integral (2) taking into account that quantity h is determined from the condition of transition of complex-conjugated roots $u_{3,4} = u_{34} \pm iw$ into real ones $u_{3,4} = u_{34} = u_*$, $w = 0$.

Thus, we have investigated the transient modes of motion of a spacecraft with a biharmonic moment characteristic at the upper section of a trajectory both in a planar and in a spatial case of motion. The analytical formulas are found for an action integral, expressed in terms of full elliptic integrals, and along the separatrices, in terms of elementary functions. Based on the latter, we determined the moments of time corresponding to the transitions between various regions of the system's phase portrait and the boundary conditions for kinematic parameters of motion. For the cases of motion, when intersecting a separatrix, the phase point may fall into various oscillation regions; the formulas are found for determining the probability of capture into any region.

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