Capture trajectories into vicinity of collinear libration points by variable electrostatic field

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Nomenclature

d = distance between primeries (planet-moon), m
G = Newtonian gravitational constant, 6.67428 · 10^{-11}, $\mathrm{m}^3 \cdot s^{-2} \cdot kg^{-1}$
k_C = Coulomb's constant, 8.99 · 10^9 $N \cdot \mathrm{m}^2 / C^2$

m_1 = mass of a planet, kg

m_2 = mass of a moon, kg

m_3 = mass of the E-body, kg

n = mean orbital rate of the space tug, rad / s

q_{orb} q_3 = charge equal for the orbiter and the E-body, C
λ_D = Debye length, m

μ = $m_2 / (m_1 + m_2)$

Φ = product $k_C q_{orb} q_3$, $\mathrm{N} \cdot \mathrm{m}^2$

Subscripts

1 = Planet

2 = Moon

3 = E-body

I. Introduction

The restricted three-body problem is a classic celestial mechanics issue. Its study made a great contribution to the theory of space dynamics and celestial mechanics. The great mathematicians Euler and Lagrange were at the origin of the solution to this fundamental problem. The three collinear Lagrange points were discovered by Euler [1] a few

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years before Joseph-Louis Lagrange discovered the remaining two [2]. In 1772, Lagrange considered the general three-body problem and demonstrated two special constant-pattern solutions, the collinear and the equilateral, for any three masses, with circular orbits. The restricted three-body problem is the basis for solving many space applications, in particular, for calculating interplanetary flights and launching spacecraft and satellites. A detailed analysis of major studies on this topic can be found in the Szebehely's textbook [3]. The book discusses the regularization of the motion equations, manifold of the states of motion, equilibrium positions, motion near these positions, application of Hamiltonian dynamics methods to the restricted problem, its periodic orbits, and quantitative aspects. A detailed numerical analysis of three-dimensional periodic halo orbits near collinear libration points in the restricted three-body problem was performed by Howell [4]. For application to the n-body problem Marchand, Howell and Wilson [5] developed efficient techniques for preliminary design of trajectory arcs in nonlinear autonomous dynamic systems in which a solution is subject to algebraic interior and exterior constraints. Woo and Misra [6] investigated the spacecraft motion in the vicinity of a binary asteroid system as the circular restricted case. The asteroids were considered as rigid bodies. Addition equilibrium points were found numerically for some special cases. Biggs and Negri [7] considered solar sail spacecraft controlled motion within the circular restricted three-body problem. The spacecraft moves in the gravitational field of the Earth and the Moon, taking into account the perturbation introduced by the solar pressure on the sail. Alessi and Sánchez [8] presented a semi-analytical approach, which is based on a perturbation procedure, for study the three-dimensional motion of a negligible mass body in the circular restricted three-body problem. In the context of the three-body problem, a relative dynamics of two spacecraft (chaser and target), flying in the vicinity of the smallest primary, is considered by Franzini and Innocenti in [9]. Catlin and McLaughlin [10] presented an investigation of the existence and nature of formation flight trajectories near the Earth–moon triangular libration points in the circular restricted three-body problem, and obtained analytical equations of motion that describe a relative dynamics within a rotating, lead-satellite-centered coordinate frame.

A new development of the classical restricted three-body problem was presented in the papers by Aslanov [11, 12], which consider the motion of a small electrostatic body (E-body) in an attractive electrostatic potential field (E-field) generated by an orbital spacecraft (orbiter) located at one of the unstable collinear libration points. In this case the functional value of the collinear libration point changes radically. The unstable libration point transforms into an attracting center. The "old" collinear libration point splits into two new unstable collinear libration points to the right and left of the "old" point, forming a new Hill's E-sphere [12]. Thus, two gravitational fields from big bodies (primaries) and the E-field with the center at the «old» collinear libration point affect the small E-body. All these fields are potential, and if the body enters, the Hill's E-sphere centered at the "old" libration point, over time it will leave it.

If it is required that the E-body reaches the vicinity of the libration point, then in addition to potential forces, a dissipative (non-potential) force must also acts. In [13] additional forces from the permanent magnets placed on the orbiter and the E-body (capsule) solve this problem, while in paper [11] this issue was not considered at all.

The purpose of the paper is to create a control law of the electrostatic charge of the orbiter, which irreversibly leads the E-body in the vicinity of the "old" collinear libration point. Solving this problem involves three stages:

- Formulation of assumptions that do not violate a physical essence of the problem, and derivation of a mathematical model convenient for further calculations.
- Development and analytical analysis of a feedback control law of the orbiter charge.
- Verification of the feedback control law by means of numerical simulation.

These three stages are devoted to three main sections of the paper.

II. Key assumptions and motion equations in polar coordinates

A. Key assumptions

We introduce acceptable assumptions that do not violate a physical essence of the study problem:

1. A circular restricted three-body problem is studied in which the two main bodies (primaries) are considered homogeneous.
2. Mass of the E-body m_3 is significantly less than the mass of any of the primaries

$$m_3 \ll m_2 < m_1 \tag{1}$$

3. In all considered cases only in-plane motion is studied.
4. The Hill's sphere is located inside Debye sphere, as it appeared was accepted in [11,12]. Note that Debye length (radius of the Debye sphere) λ_D is an important parameter because the Electrostatic field rapidly decreases beyond this length by the Debye shielding effect. However, the Debye length near the collinear libration points is usually unknown. Given in the literature, for example in [15], only approximate values of the Debye length for the Stickney crater, located on Phobos' surface directly under the L1 libration point at a distance of approximately 3.5 km. Depending on Mars local time, this parameter ranges from 13 m to 47 m.
5. A simple sphere-spherical model of electrostatic force is employed as an electrostatic model of the orbiter and E-body, which is obtained using assumptions and equations from the paper by Jasper and Schaub [16]. The orbiter and E-body are modeled as spheres with radii R_1 and R_2, respectively. If the distance between

the bodies (R) is assumed to be much greater than the given radii $R \gg R_1, R_2$, then the simpler equations take place

$$q_{orb} = \frac{R_{orb} V_{orb}}{k_c}, \quad q_3 = \frac{R_3 V_3}{k_c} \tag{2}$$

where q_{orb}, q_3 are charges of the orbiter and E-body, respectively; V_{orb}, V_3 are voltage of the orbiter and E-body, respectively; and $k_c = 8.99 \cdot 10^9 N \cdot \mathrm{m}^2 / C^2$ is the Coulomb constant. In this case an electrostatic force F between the orbiter and E-body is:

$$F = k_c \frac{q_{orb} q_3}{R^2} = \frac{\Phi}{R^2} \tag{3}$$

where $\Phi = k_c q_{orb} q_3$.

B. Motion equations in polar coordinates

Consider the equations of the E-body planar motion in the Local-Vertical-Local-Horizontal frame Oxy within the scope of the classical restricted three-body problem [3,14]

$$\ddot{x} = \frac{\partial W}{\partial x} + n^2 x + 2n\dot{y} + \Phi \frac{x-a}{R^3} \tag{4}$$

$$\ddot{y} = \frac{\partial W}{\partial y} + n^2 y - 2n\dot{x} + \Phi \frac{y}{R^3} \tag{5}$$

where

$$W(x,y) = G\left(\frac{m_1}{\sqrt{(x+d\mu)^2 + y^2}} + \frac{m_2}{\sqrt{(x-d(1-\mu))^2 + y^2}}\right) \tag{6}$$

$$\mathbf{R} = \overline{M_1 M_3} - \overline{M_1 L_1} = (X, Y) \tag{7}$$

$$\mathbf{F} = \Phi \frac{\mathbf{R}}{R^3}, \tag{8}$$

where $\mu = \dfrac{m_2}{m_1 + m_2}$, d is the distance between Mars and Phobos, a is the abscissa of the L_i libration point $i = 1,2,3$, m_1 is mass of a planet, m_2 is mass of a moon, $\mathbf{F}$ is the Coulomb force as a vector, X, Y are the coordinates of the frame $L_i XY$. Points M_i are shown in Fig. 1.

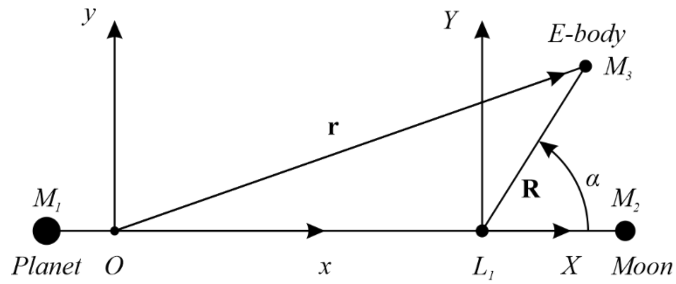

Fig.1 The frame L_iXY and the polar coordinates R,α

Thus, Eqs. (4) and (5) describe the flight of the E-body under the action of the attractive electrostatic force (8) and the gravitational influence of the uniformly rotating planet-moon system. Since we study the motion of the E-body relative to the orbiter, which is in the L_i libration point $(i=1,2,3)$, it makes sense to pass from the frame Oxy to the frame L_iXY by changing the variables (Fig. 1). Position of the E-body M_3 relative to the L_i libration points in a polar reference frame (R,α) is defined by substituting the variables

$$x = a + R\cos\alpha,\ \ y = R\sin\alpha \tag{9}$$

where a is the abscissa of the L_i libration point $(i=1,2,3)$, which are the roots of Eq. (4) at no the Coulomb force and as shown in [14] when: $y=0,\ \dot{y}=0,\ \ddot{x}=0$. Eqs. (4) and (5) in the polar reference frame (Fig. 1) are written as

$$\ddot{R} = \frac{\partial U}{\partial R} + R(n+\dot{\alpha})^2 \tag{10}$$

$$\ddot{\alpha} = \frac{\partial U}{\partial \alpha} - 2\frac{\dot{R}}{R}(n+\dot{\alpha}) \tag{11}$$

The potential of Eqs. (4) and (5) is written as

$$U = G\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \frac{\Phi}{m_3 R} + \frac{1}{2}(nr)^2 \tag{12}$$

where the distance between the mass center of the primaries and the E-body is

$$r = \sqrt{R^2 + 2R(a - p\mu)\cos\alpha + (a - p\mu)^2} \tag{13}$$

the distance between the primaries 1 and the E-body

$$r_1 = \sqrt{a^2 + R^2 + 2aR\cos\alpha} \tag{14}$$

and the distance between the primaries 2 and the E-body is

$$r_2 = \sqrt{(d-a)^2 - 2(d-a)R\cos\alpha + R^2} \tag{15}$$

The charges of the orbiter and the E-body are constant and have the opposite sign, therefore

$$\Phi = k_C q_{orb} q_{cont} = const < 0 \tag{16}$$

Eqs. (10) and (11) correspond to the energy integral (total energy) per unit mass

$$E = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\alpha}^2\right) - U = const \tag{17}$$

The negative doubled total energy per unit mass in the rotating Cartesian frame Oxy is a first integral, called the Jacobi integral [3,14] which is written as

$$J = -2E = const\,, \tag{18}$$

As shown in Fig. 2, the attractive E-field splits the L_i ($i = 1,2,3$) collinear libration point into two new collinear points L_{i-}, L_{i+}, and the greater the potential (12) and the Debye length the greater the distance between the new libration points and the ''old'' original libration point [11,12]. The E-Hill's sphere is the region in which it dominates the attraction of the E-body. The boundary of the E-Hill's sphere is the surface of zero velocity and for the planar case represents the cross section of zero relative velocity

$$U(R,\alpha) = const \tag{19}$$

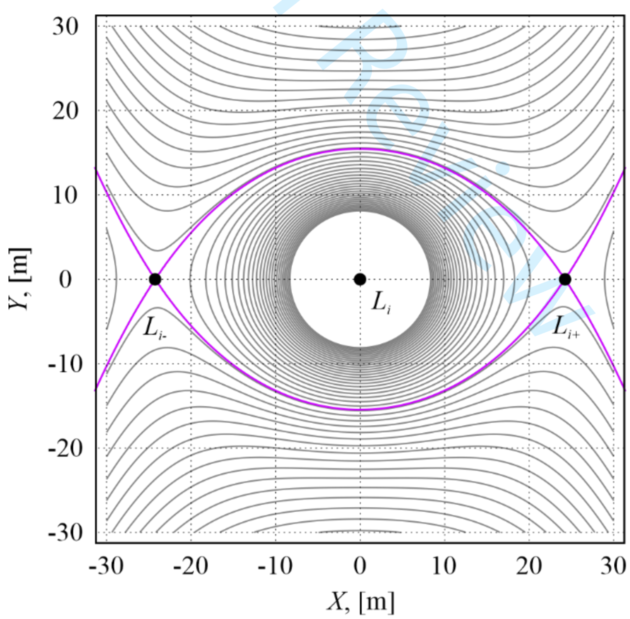

Fig. 2 Splitting of the unstable Mars–Phobos L_1 libration point to two the unstable L_{1-} and L_{1+} points, the "old" L_1 libration point, the zero-relative-velocity cross section (blue) for $\Phi = -0.066741\,\mathrm{Nm}^2$ and $m_3 = 10\,kg$

The new collinear libration points L_{1-} and L_{1+} presented in Fig. 2, caused by the E-field, lie on the axis L_1X, are restricted by the Debye sphere and can be found as solutions to the following equation

$$\left(\frac{\partial U}{\partial R}\right)_{/\alpha=0,\pi} = 0 \quad (20)$$

The coordinates of the new libration points in the polar reference frame R,α are as follows

$$L_{1+}:\ R = 24.220\,m,\ \ \alpha = 0 \quad (21)$$

$$L_{1-}:\ R = 24.243\,m,\ \ \alpha = \pi \quad (22)$$

III. Choice and analytical analysis of a feedback charge control law

In this section, a feedback charge control law is introduced to stabilize separation distance rate between the E-body and the orbiter, which is at the L_i "old" collinear point. According to Eqs. (10) and (11), only potential forces act on the E-body, so if the body crosses the boundaries of the E-Hill's sphere from outside, it means that its total energy is greater than the potential energy at one of the unstable points L_{i-} or L_{i+}. In this case, after some time, the E-body can leave the E-Hill's sphere. An irreversible motion of the body from the Hill sphere boundary to the E-field center (the "old" collinear libration point) can occur only due to the action of dissipative forces, i.e. forces that reduce the total energy of the body. It should be kept in mind that the only means of control is the value of the orbiter charge and the electrostatic force acts on the E-body in radial direction, i.e. along the radius R in the polar reference frame R,α .

Assume the charge control law with feedback of the separation distance rate as

$$q_{orb} = q_0\left(1+\varepsilon\frac{\dot{R}}{Rn}\right) \quad (23)$$

where q_0 is the initial value of the orbiter charge at reaching the vicinity of the E-Hill's sphere by the E-body, ε is the dimensionless control coefficient. If the orbiter charge is variable, the product Φ will also be variable in the motion Eqs. (10), (11) and (12)

$$\Phi = \Phi_0\left(1+\varepsilon\frac{\dot{R}}{Rn}\right) \quad (24)$$

For the attracting E-field, the charges of the orbiter and the E-body have the opposite sign, therefore

$$\Phi_0 = k_C q_0 q_{cont} = const < 0 \quad (25)$$

Differentiating the total energy (17) of the E-body by Eqs. (10), (11) and (12), taking into account the orbiter charge (23), gives the rate of change of the total energy as

$$\frac{dE}{dt} = \varepsilon \frac{\Phi_0}{m_3 R^3 n} \dot{R}^2 < 0 \tag{26}$$

It follows from Eqs. (20) and (26) an obvious fact that when orbiter charge is controlled by the feedback law (23), the total energy of the E-body decreases over time. However, the feedback law (23) has an effect on the E-body until the body motion turns to a uniform rotation around the "old" libration point inside the E-Hill's sphere, i.e.

$$\dot{R} \to 0,\ R \to const,\ \dot{\alpha} \to const \tag{27}$$

Note, the feedback law (23) may have a more common kind, which will lead to a reduction of the total energy (17), so, for example

$$q_{orb} = q_0 \left[1 + \varepsilon \left(\frac{\dot{R}}{Rn} \right)^j \right] \tag{28}$$

where $j = 1,3,5...$ is a positive odd integer. In this case the derivative of the total energy can be written as

$$\frac{dE}{dt} = \varepsilon \frac{\Phi_0}{m_3 R^{2+j} n^j} \dot{R}^{j+1} < 0 \tag{29}$$

IV. Numerical simulation

This section illustrates the effectiveness of the proposed control law (23) using numerical simulation of Eqs. (10) and (11) as an example of the L_1 libration point in the Mars-Fobos system, in which the orbiter is located. Taking into account Eqs. (23) and (2) the orbiter's voltages should be changed according to the control law

$$V_{orb} = V_0 \left(1 + \varepsilon \frac{\dot{R}}{Rn} \right) \tag{30}$$

where V_0 is the initial constant voltage. For the numerical simulation, the following parameters of the orbital apparatus and E-body are accepted

$$R_{orb} = 3.0\,m,\ V_0 = 20\,kV, R_3 = 0.5\,m,\ V_3 = 20\,kV,\ m_3 = 10\,kg \tag{31}$$

Then by virtue of Eqs. (2) and (16) the initial value of the product is

$$\Phi_0 = -0.066741\,\mathrm{Nm}^2 \tag{32}$$

and the coordinates of the unstable libration points

$$L_{1+}:\ R = 24.220\,m,\ \ \alpha = 0$$

$$L_{1-}:\ R = 24.243\,m,\ \ \alpha = \pi$$

Consider four trajectories of the E-body, which start at four different points as shown in Figs. 3 and 5

$$\alpha_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \tag{33}$$

Other initial conditions for numerical integration of the equations are taken the same for all initial points

$$R_0 = 27.0\,m\ \ \dot{R}_0 = -0.01\,m\,/\,s\,, \dot{\alpha}_0 = 0 \tag{34}$$

If talking about getting the E-body inside E-Hill's sphere, then outside this sphere, the total energy (17) of the E-body should be greater than the potential energy on the sphere’s shell (12) and, therefore, at the saddle points L_{1-} and L_{1+}. Only in this case, the E-body trajectory can cross the sphere's shell. The total energy at the initial points of the trajectories (33), (34) exceeds the potential energy corresponding the unstable points of libration L_{1-} and L_{1+}, i.e.

$$E\ \ R_0, \dot{R}_0, \alpha_0, \dot{\alpha}_0\ \ > -U\ \ R_*, \alpha_* \tag{35}$$

where the coordinates R_* and α_* of the points L_{1-} and L_{1+} are determined by Eqs. (21) and (22).

Figs. 3 and 5 show the trajectory capture in the vicinity of the L_1 “old” libration point by controlling the orbiter voltages (30) the four trajectories of the E-body (33) for two values of the control coefficient $\varepsilon = 0.01,\ 0.1$. Figs. 4 and 6 depict the profile of the orbiter voltage corresponding to the trajectories shown in Figs. 3 and 5. The voltages for the trajectories starting at opposite points are the same.

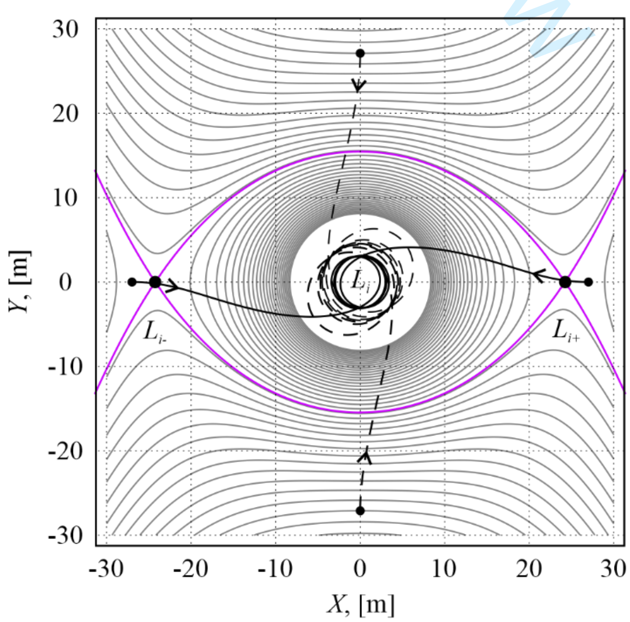

Fig. 3 Capturing the four trajectories of the E-body (33) in the vicinity of the L_1 "old" libration point by controlling the orbiter voltages (30) for the control coefficient $\varepsilon = 0.01$

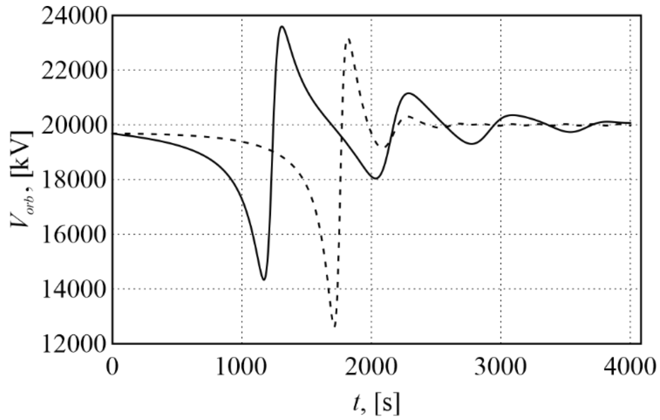

Fig. 4 The orbiter voltages for the four controlled trajectories of the E-body (33) by means of the law(30) for the control coefficient $\varepsilon = 0.01$ (dotted line $\alpha_0 = \frac{\pi}{2}, \frac{3\pi}{2}$; solid line $\alpha_0 = 0, \pi$)

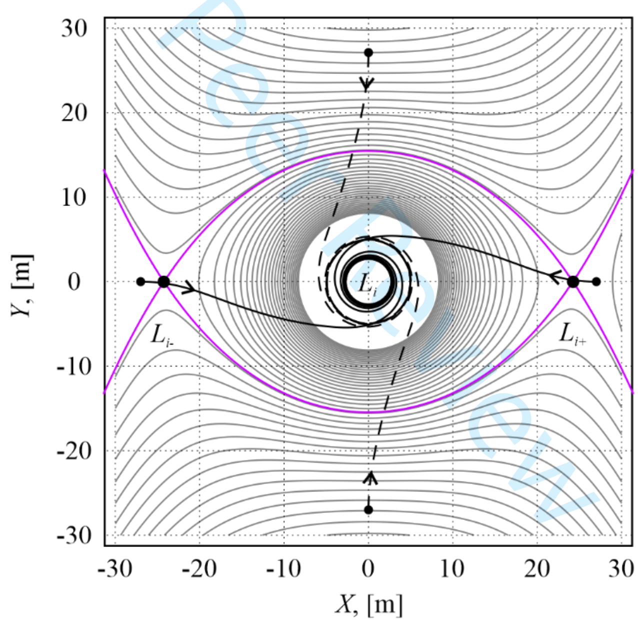

Fig. 5 Capturing the four trajectories of the E-body (33) in the vicinity of the L_1 "old" libration point by controlling the orbiter voltages (30) for the control coefficient $\varepsilon = 0.1$

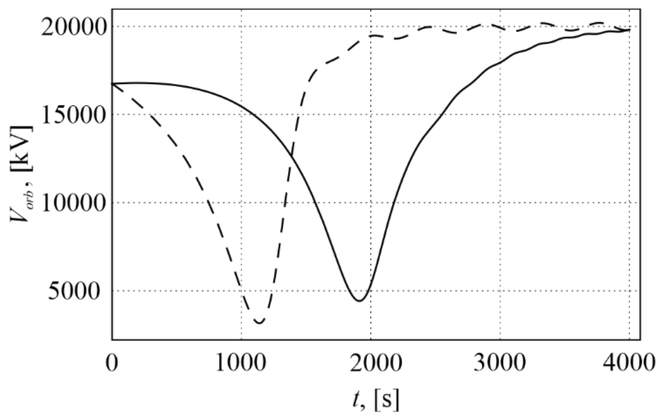

Fig. 6 The orbiter voltages for the four controlled trajectories of the E-body (33) by means of the law(30) for the control coefficient $\varepsilon = 0.1$ (dotted line $\alpha_0 = \frac{\pi}{2}, \frac{3\pi}{2}$; solid line $\alpha_0 = 0, \pi$)

As shown in Figs. 3 and 5, over time, all trajectories of the E-body go to a slow steady rotation around the L_1 "old" libration point at a distance of no more than 5 m. In the first case (Fig. 3), the relative velocity of the E-body is approximately 0.05 m/s and in the second case (Fig. 5), it is less than 0.04 m/s.

Note that in the first case (Fig. 4) the orbiter voltage reaches 24 kV, which is 20% higher than the nominal voltage ($V_0 = 20\,kV$). In this sense, the second case (Fig. 6) is preferable to the first, because the orbiter voltage slightly exceed the nominal value of 20 kV.

If the orbiter voltages are not controlled when the control coefficient $\varepsilon = 0$ in the control law (30), and the nominal value ($V_{orb} = V_0$) is maintained, the four trajectories leave the E-Hill's sphere without any rotation around the L_1 "old" libration point as shown in Figs. 7.

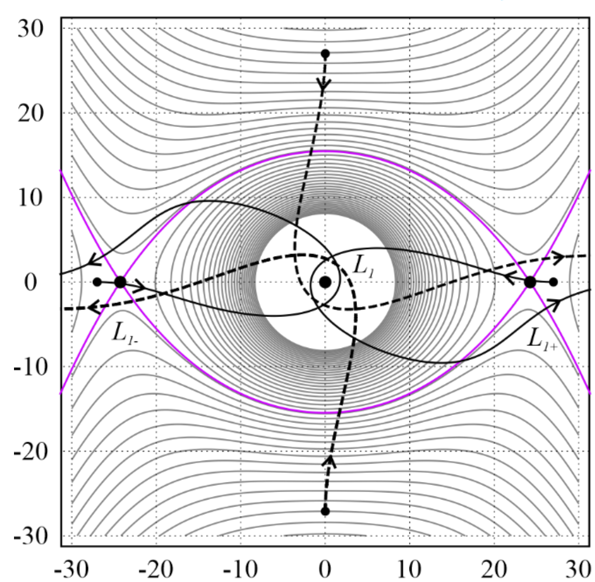

Fig. 7 The four trajectories of the E-body (33) in the vicinity of the L_1 "old" libration point without the control ($\varepsilon = 0$)

This section confirms the possibility to use the proposed orbiter voltage control law for E-body capture (30), at the same time it is clear that detailed studies, taking into account limitations on the rate of change and on the magnitude of the orbiter voltage and other limitations and other restrictions should be performed in future works.

V. Conclusions

This paper is a development of the work [11,12] on the splitting of collinear libration points by means of a stationary artificial electrostatic field, which now considers a variable E-field. The new main results and conclusions of the paper can be summarized as follows:

- The feedback law for controlling the orbiter charge located at one of the collinear libration points was proposed. It is analytically proved that the time derivative of the energy of relative motion of the E-body at a positive sign of the control coefficient ($\varepsilon > 0$) is negative. This control law reduces the total energy of relative motion of the electrostatic body and leads the E-body to a steady rotation around the libration point over time. The analytical results and effectiveness of the proposed voltage control law (30) were confirmed by numerical simulations of E-body motion.

- Numerical simulations have shown that if the E-body begins to move outside the Hill E-sphere with 27 m distance from the L_1 libration point, and, as a result of control of the voltage magnitude by the law (30), over time the E-body motion transitions to a nearly regular rotation at a distance no more than 5 m from the L_1 libration point with a relative velocity not exceeding 0.05 m/s. In this case, the voltage magnitude of the orbiter required for control did not exceed 20 kV for the control coefficient $\varepsilon = 0.1$.

In future studies on the capture of the E-body in the vicinity of the collinear libration point, other charge control laws will be proposed. A study will be carried out to determine the region of possible motions of the E-body near the E-Hill's sphere, for which the E-body capture can be realized using the proposed control law (30). In addition, the proposed approach may be useful as other planet-moon systems and for other collinear libration points.

Funding Sources

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